Worksheet 9

Sections 306 and 310
MATH 54

September 20, 2018

Exercise 1. Let W be the union of the first and third quadrants of the plane. That is: let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$.

(a) If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

(b) Can you find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that their sum is not in W.

(c) Is W a vector space?

(a). Yes. W is the set of $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x \geq 0$ and $y \geq 0$ or $x \leq 0$ and $y \leq 0$ multiplying by a constant c. doesn't change the fact that x, y have the same sign.

(b) $\begin{bmatrix} -1 \\ -5 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ which is not in W.

(c) No, since it doesn't satisfy the property in part b.

Exercise 2. For each of the following sets, either use an appropriate theorem to show that the given set is a vector space, or find an specific example to the contrary.

$$\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 5r - 1 = s + 2t \right\}$$

No! $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in this set since $5 \cdot 0 - 1 \neq 0 + 0$.

A vector space must contain the 0 vector.

$$\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a + 3b = c \\ b + c + a = d \end{matrix} \right\}$$

Yes!

We can rewrite the equations as $a + 3b - c = 0$, $b + c + a - d = 0$

Thus we see that this is the nullspace of the matrix

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

A null space of a matrix is always a vector space.

Exercise 3. Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. Is $\mathbf{w}$ in Col A? Is it in Nul A?

Col A.

To check if $\vec{w}$ is in col A, we check if $A\vec{x} = \vec{w}$ has a solution. We can do this by reducing the following augmented matrix.

$$\begin{bmatrix} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(I skipped some steps)

This is inconsistent, so $A\vec{x} = \vec{w}$ doesn't have a solution.

So $\vec{w}$ is not in col A.

$A\vec{w} = \begin{bmatrix} -9 \\ 8 \\ 4 \end{bmatrix}$ This is not $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $\vec{w}$ is not in the null space.

Exercise 4. Determine which of the following sets are bases for $\mathbb{R}^3$. Justify your answers.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \begin{bmatrix} -4 \\ -5 \\ 6 \end{bmatrix}$$

Discuss with your group: Do you think that a set of two vectors can form a basis for $\mathbb{R}^3$? Why or why not? (We will discuss the idea of dimension soon, get excited!!)

(a). Not a basis! Not linearly independent.

(b) A basis. We can row reduce the matrix $\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$ to the identity $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By the invertible matrix theorem, this has lin ind. columns and these columns span $\mathbb{R}^3$.

(c). Not a basis. The vectors do not span $\mathbb{R}^3$. To see this, we look at $\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \vec{x} = \vec{b}$.

$\begin{bmatrix} 1 & -4 & b_1 \\ 2 & -5 & b_2 \\ -3 & 6 & b_3 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & c_3 \end{bmatrix}$ which is not always consistent.

It turns out that all bases of $\mathbb{R}^3$ have 3 vectors, we will talk about this later!

Exercise 5. Let W be a vector space. Use the axioms of a vector space to show that $0\mathbf{u} = \mathbf{0}$ for every vector u in W.

$$0\vec{u} = (0+0)\vec{u} = 0\vec{u} + 0\vec{u}$$

since $0+0=0$ — by distributive property.

We also know that every vector $\vec{v}$ has an additive inverse $-\vec{v}$. So $0\vec{u}$ has an additive inverse $-0\vec{u}$.

We add this to both sides:

$$0\vec{u} + (-0\vec{u}) = (0\vec{u} + 0\vec{u}) + (-0\vec{u}).$$

Using associativity, we can rearrange the right hand side:

$$0\vec{u} + (-0\vec{u}) = (0\vec{u}) + (0\vec{u} + (-0\vec{u})).$$

Note that $0\vec{u} + (-0\vec{u}) = 0$ by def of additive inverse.

So we are left with

$$0 = 0\vec{u}$$

as desired.