Worksheet 8

Sections 306 and 310
MATH 54

September 18, 2018

Exercise 1. Determine the values of s such that the system has a unique solution. Use Cramer's rule to describe the solutions in terms of s.

$$3sx_1 + 5x_2 = 3$$

$$12x_1 + 5sx_2 = 2$$

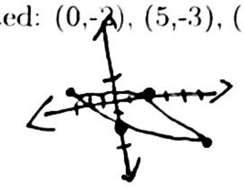
We rewrite this as $\begin{bmatrix} 3s & 5 \\ 12 & 5s \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. This has exactly one solution if and only if $\begin{bmatrix} 3s & 5 \\ 12 & 5s \end{bmatrix}$ is invertible, which is the case if and only if $\begin{vmatrix} 3s & 5 \\ 12 & 5s \end{vmatrix} = 15s^2 - 60 = 15(s+2)(s-2) \neq 0$.

So this has a unique solution if and only if $s \neq \pm 2$.

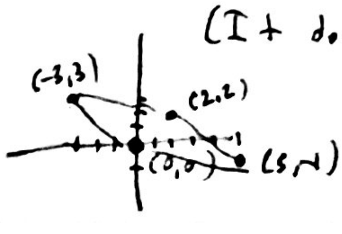
By cramers rule, $x_1 = \dfrac{\begin{vmatrix} 3 & 5 \\ 2 & 5s \end{vmatrix}}{\det A} = \dfrac{15s-10}{15s^2-60}$

$$x_2 = \frac{\begin{vmatrix} 3s & 3 \\ 12 & 2 \end{vmatrix}}{\det A} = \frac{6s-36}{15s^2-60}.$$

Exercise 2. Find the area of a parallelogram whose vertices are listed: (0,-2), (5,-3), (-3,1), (2,0).

We first draw the parallelogram  (this is a parellogram I promise :))

In order to use the determinant formula for area, we have to move it so that one of the vertices is at the origen. One way to do this is shifting it up by 2. (It doesn't matter which vertex you choose to move to the origin.



The two vertices adjacent to the origin are (5,-1) and (-3,3). So by the determined formula for area we have: Area $= \left|\det \begin{bmatrix} 5 & -3 \\ -1 & 3 \end{bmatrix}\right| = 12$.

Exercise 3. Find the area of a triangle whose vertices are $(0,0)$, (v_1, v_2), (w_1, w_2)

From our parallelogram formula, we know that the area of the following parallelogram is $\left|\det\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\right|$

The triangle we are interested in is exactly half this area.

So the area is $\frac{1}{2}\left|\det\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\right|$

Exercise 4. Determine if the following sets are polynomials are subspaces of the space $\mathbb{P}_3$ of polynomials in t of degree at most 3.

a. All polynomials of the form $a + t^2$, where a is in $\mathbb{R}$.

b. All polynomials p in $\mathbb{P}_3$ such that $p(0) = 0$.

(a). This is not a vector space. One way to show this, that the 0 polynomial, $p_0(t) = 0 + 0t + 0t^2 + 0t^3 = 0$ is not in the set, since the coefficient of t^2 in p_0 is not $\underline{1}$. Since every subspace of $\mathbb{P}_3$ contains the 0-polynomial, this set cannot be a subspace.

(b) This is a subspace. To show this, we show that the 3 properties of a subspace hold.

(1) We need to show that the 0-polynomial $p_0(t) = 0$ is in the set. This is true since $p_0(0) = 0$.

(2) Let $p_1(t)$, $p_2(t)$ be in the set. We need to show that $p_1 + p_2(t)$ is in the set. Since p_1, p_2 are in the set, $p_1(0) = 0$ and $p_2(0) = 0$.

Thus, $p_1 + p_2(0) = p_1(0) + p_2(0) = 0$.

as desired.

(3) Let p_1 be in the set, and c be a scalar. We need to show that cp_1 is in the set. We know that $p_1(0) = 0$. Thus,

$cp_1(0) = c \cdot 0 = 0$ as desired.

Exercise 5. Let W be the set of all vectors of the form $\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}$, where b, c can be any real numbers. Find $\mathbf{u}, \mathbf{w}$ such that W is the span of $\mathbf{u}, \mathbf{w}$. Is W a subspace of $\mathbb{R}^3$?

W is the set of all vectors that can be written in the form $\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}$ which can be rewritten as $b\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. (Think about when you wrote solutions in parametric form). By the definition of span, the set of all vectors of this form is $\text{span}\left\{\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$.

The span of a set of vectors is always a subspace (see a theorem in the book) so W is a subspace of $\mathbb{R}^3$.

Exercise 6. Find an explicit description of Nul(A) by listing vectors that span the null space:

$$A = \begin{bmatrix} 1 & 6 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

How many entries do vectors in the null space have? How many entries to vectors in the column space have?

The null space is the set of all $\vec{x}$ such that $A\vec{x} = \vec{0}$.
We solve this by row reducing the following augmented matrix:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 6 & -4 & -3 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-6R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 8 & -9 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = -8x_3 + 9x_4 - x_5$
$x_2 = 2x_3 - x_4$, Putting this in parametric vector form, we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3\begin{bmatrix} -8 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 9 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So $\text{Nul}(A) = \text{span}\left\{\begin{bmatrix} -8 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ ← each vector has 5 entries

By definition, $\text{Col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ ← each vector has 3 entries