Worksheet 7

Sections 306 and 310
MATH 54

September 13, 2018

Exercise 1. Let T be a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 10 & 3 \end{bmatrix}.$$

Is T one-to-one? onto? Discuss what this means in your own words with your group.

We first row-reduce this: $\begin{bmatrix} 1 & 2 & 0 \\ 5 & 10 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

There is a free variable, so T is not one-to-one.

However, the columns of A span $\mathbb{R}^2$ $\left(\begin{bmatrix} 1 & 2 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \end{bmatrix} \text{ will always be consistent} \right)$

so T is not onto.

Exercise 2. State the row operation shown below and describe how it affects the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$$

Actually compute the determinants, don't use Theorem 3.

We added kR_2 to R_1.

The determinant of the left hand side is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

The determinant of the right hand side is $\begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} =$

$ad + \cancel{kcd} - bc - \cancel{kdc} = ad - bc.$

So the determinant is not affected by this row operation.

Exercise 3. Find the determinant by row reduction to echelon form. (It is ok (and encouraged!) to use Theorem 3 for this exercise.

$$\begin{vmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & -1 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & -3 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{vmatrix}$$

$$= 3\,(-8) = -24$$

the det of a triangular matrix is the product of the diagonals.

Exercise 4. Suppose that we already know that:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

Compute the following determinant:

$$\begin{vmatrix} -5d+g & -5e+h & -5f+i \\ a & b & c \\ g & h & i \end{vmatrix}$$

$$\begin{vmatrix} -5d+g & -5e+h & -5f+i \\ a & b & c \\ g & h & i \end{vmatrix} = \begin{vmatrix} -5d & -5e & -5f \\ a & b & c \\ g & h & i \end{vmatrix} =$$

$$-5\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = 5\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5\cdot 7 = 35.$$

Exercise 5. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det(5A) = 5\det(A)$? Let A be a $n \times n$ matrix and let k be a scalar. Find a formula for $\det(kA)$ in terms of k and $\det(A)$.

$\det A = 6 - 4 = 2.$

$\det 5A = \begin{vmatrix} 15 & 5 \\ 20 & 10 \end{vmatrix} = 150 - 100 = 50.$

So $5 \det A \neq \det 5A$.

It turns out that $\det(kA) = k^n \det(A)$.

We can we the fact that $\det(AB) = \det(A)\det(B)$ to show this.

Note that $kA = \begin{bmatrix} k & & 0 \\ & k & \\ 0 & & \ddots \; k \end{bmatrix} A.$

k's on diagonal,
0's everywhere else

So $\det(kA) = \det \begin{bmatrix} k & & 0 \\ & k & \\ 0 & & \ddots \; k \end{bmatrix} \det A = k^n \det(A).$

the determinant of a
diagonal matrix is the
product of the diagonals.