Worksheet 6

Sections 306 and 310
MATH 54

September 11, 2018

Exercise 1. Solve the system using matrix inverses!

$$8x_1 + 5x_2 = -9$$

$$-7x_1 - 5x_2 = 11$$

We first write this as a matrix equation: $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix} \vec{x} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$.

We can tell that $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$ is invertible since its determinent is $-5 \neq 0$.

We now find the inverse:

For a 2×2 matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ There is a formula for the inverse:

$\frac{1}{\det B} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. So $A^{-1} = -\frac{1}{5} \begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -7/5 & -\frac{8}{5} \end{bmatrix}$. Multiplying both sides of $A\vec{x} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$, we get $\vec{x} = A^{-1} \begin{bmatrix} -9 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{bmatrix} \begin{bmatrix} -9 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

Exercise 2. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is an invertible $n \times n$ matrix. show that $B = C$.

We know that $(B-C)D = 0$. $(\star)$

Since D is invertible, D has an inverse, D^{-1}.

We multiply both sides of $(\star)$ by D^{-1} on the right side:

$(B-C) D D^{-1} = 0 D^{-1}$.

Note that $DD^{-1} = I_n$, and $0D^{-1} = 0$.

So we are left with $B - C = 0$, which rearranges to $B = C$.

Exercise 3. Determine which of the matrices are invertible. Justify your answers, but try using as few calculations as possible :)

$$\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix} \qquad \begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Not invertible! The determinant is $(-4)(-9) - 6 \cdot 6 = 0$. See Thm 4 on page 173.

Not invertible! There is a column of all 0's, so the columns do not form a lin. ind. set. See part (e) of Thm 8 on page 114.

Invertible! This is already in row-echelon form, and we can see there are 4 pivot positions. See part (c) of Thm 8 on page 114.

Exercise 4. Is it possible for a 5×5 matrix to be invertible when its columns do not span $\mathbb{R}^5$? Why or why not?

It is not possible.

By part (h) of Thm. 8 on page 114, a 5×5 matrix is invertible if and only if its columns span $\mathbb{R}^5$.

Exercise 5. Compute the following determinant by cofactor expansion.

$$\begin{vmatrix} 3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 2 & 6 & 0 & 3 \\ 3 & -8 & -3 & 4 \end{vmatrix}$$

For a 4x4 matrix, the pattern of signs is:

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Since the third column has a lot of 0's, we will use that for our cofactor expansion. The determinant is

$$0\begin{vmatrix} 2 & -2 & 0 \\ 2 & 6 & 3 \\ 3 & -8 & 4 \end{vmatrix} - 0\begin{vmatrix} 3 & 0 & 0 \\ 2 & 6 & 3 \\ 3 & -8 & 4 \end{vmatrix} + 0\begin{vmatrix} 3 & 0 & 0 \\ -2 & 2 & 0 \\ 3 & -8 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 3 & 0 & 0 \\ 2 & -2 & 0 \\ 2 & 6 & 3 \end{vmatrix} =$$

$$3\begin{vmatrix} 3 & 0 & 0 \\ 2 & -2 & 0 \\ 2 & 6 & 3 \end{vmatrix} = 3\left[3\begin{vmatrix} -2 & 0 \\ 6 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & -2 \\ 2 & 6 \end{vmatrix}\right] =$$

We now cofactor expand the first row.

$$3 \cdot 3\begin{vmatrix} -2 & 0 \\ 6 & 3 \end{vmatrix}] = 3 \cdot 3(-6-0) = -54$$

Exercise 6. Let A, B be $n \times n$ matrices. Show that if AB is invertible, then so is A.

Since AB is invertible, there exists W such that

$$(AB)W = I_n.$$

Since matrix multiplication is associative, we can rewrite this as

$$A(BW) = I_n.$$

So by part k of Thm 8 on page 114,

A is invertible.

Note: Don't use thm 6b on page 107, since in order to use this you have to already know that A and B are invertible.