Worksheet 4

Sections 306 and 310
MATH 54

September 4, 2018

Exercise 1. Determine if each set of ~~matrices~~ vectors. is linearly independent.

(a) $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

(c). $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 343 \\ 454 \\ 55 \\ -45 \\ 67 \end{bmatrix}$

(a). We check to see whether there are nontrivial solutions to $x_1\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The augmented matrix is

$$\begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & -8 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rearrange rows}} \begin{bmatrix} 2 & -8 & 1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

There are no free variables, so only the trivial solution exists. So these vectors are $\boxed{\text{linearly independent}}$

(b) This set is $\boxed{\text{linearly dependent,}}$ since the second vector is a scalar multiple of the first.

(c). There exists a nontrivial linear combination that sum to 0:

$$1\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + 0\begin{bmatrix} 343 \\ 454 \\ 55 \\ -45 \\ 67 \end{bmatrix} = 0$$

So the set is $\boxed{\text{linearly dependent.}}$

Exercise 2. For which values of h are the following vectors linearly dependent? Justify your answer!

$$\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -9 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$$

We want to see for which h does $x_1 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -9 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ have nontrivial solutions. We reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{array}\right] \underset{3R_1+R_3\to R_3}{\overset{-5R_1+R_2\to R_2}{\rightsquigarrow}} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -19 & h-15 & 0 \\ 0 & 12 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & -19 & h-15 & 0 \end{bmatrix} \xrightarrow{R_2/12} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -19 & h-15 & 0 \end{bmatrix}$$

$$\xrightarrow{15R_2+R_3\to R_3} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h-15 & 0 \end{bmatrix}$$

x_1 and x_2 both cannot be free variables.

x_3 is a free variable if and only if $h-15=0 \to h=15$

So the above vectors are linearly dependent if and only if $h=15$.

Exercise 3. Determine the possible row echelon forms of a 2×2 matrix with linearly dependent columns.

The possible row-echelon forms of a 2×2 matrix are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & \star \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & \star \\ 0 & \blacksquare \end{bmatrix}.$$

If we view these as coefficient matrices A of the augmented matrix $\begin{bmatrix} A & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix}$, $\begin{bmatrix} \blacksquare & \star & 0 \\ 0 & \blacksquare & 0 \end{bmatrix}$ is the only one that has no free variables.

So $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & \star \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$ are the possible row echelon forms for a 2×2 matrix with linearly dependent columns.

Exercise 4. For each pair $T, \mathbf{b}$, find a vector whose image under T is $\mathbf{b}$. Is this vector unique?

$$T = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Another way to phrase this question is: "Find an $\vec{x}$ such that $T\vec{x} = \vec{b}$. Is this $\vec{x}$ the only solution to the matrix equation?"

The augmented matrix of $T\vec{x} = \vec{b}$ is:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{2R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right] \xrightarrow{\frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[2R_3+R_1 \to R_1]{-2R_3+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \text{So } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

is mapped to $\vec{b}$ by T. Since there are no free variables, this is the only such vector.

The augmented matrix of $T\vec{x} = \vec{b}$ is:

$$\left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{array}\right] \xrightarrow{3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -8 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{8}R_2} \left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{5R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

So any $\vec{x}$ of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is mapped to $\vec{b}$. So $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is such a vector, and it is not unique.

Exercise 5. Describe geometrically what the following linear transformation T does. It may be helpful to plot a few points and their images!

$$T = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .5x_1 \\ x_2 \end{bmatrix}$

So T contracts the first coordinate of a point by $\frac{1}{2}$, and preserves the second coordinate.

Exercise 6. Let $\mathbf{e_1} = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix}0\\1\end{bmatrix}$, $\mathbf{y_1} = \begin{bmatrix}1\\8\end{bmatrix}$ and $\mathbf{y_2} = \begin{bmatrix}-2\\4\end{bmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e_1}$ to $\mathbf{y_1}$ and $\mathbf{e_2}$ to $\mathbf{y_2}$. What is the image of $\begin{bmatrix}x_1\\x_2\end{bmatrix}$?

We know that $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\8\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-2\\4\end{bmatrix}$.

So $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T\left(x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}\right) = x_1 T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + x_2 T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = x_1\begin{bmatrix}1\\8\end{bmatrix} + x_2\begin{bmatrix}-2\\4\end{bmatrix}$.

$\boxed{\text{by the definition of a lin trans.}}$

Exercise 7. Show that $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_2\\x_1\end{bmatrix}$ is a linear transformation.

In order to show that T is a linear transformation we need to show that $T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$ for any $\vec{u}, \vec{v}$ in $\mathbb{R}^2$, a, b in $\mathbb{R}$.

Let $\vec{u} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$, $\vec{v} = \begin{bmatrix}y_1\\y_2\end{bmatrix}$. Then:

$T(a\vec{u} + b\vec{u}) = T\left(a\begin{bmatrix}x_1\\x_2\end{bmatrix} + b\begin{bmatrix}y_1\\y_2\end{bmatrix}\right) = T\left(\begin{bmatrix}ax_1 + by_1\\ax_2 + by_2\end{bmatrix}\right) =$

$\begin{bmatrix}ax_2 + by_2\\ax_1 + bx_1\end{bmatrix} = a\begin{bmatrix}x_2\\x_1\end{bmatrix} + b\begin{bmatrix}y_2\\y_1\end{bmatrix} = aT\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) + bT\left(\begin{bmatrix}y_1\\y_2\end{bmatrix}\right) =$

$aT(\vec{u}) + bT(\vec{v})$, as desired.

So T is indeed a linear transformation.