Worksheet 3

Sections 306 and 310
MATH 54

August 30, 2018

Exercise 1. Do the following vectors span $\mathbb{R}^3$?

$$\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$$

Theorem 4 on pg. 37 of the book says the columns of a matrix A span $\mathbb{R}^n$ iff A has a pivot position in every row.
Let's find the pivot positions of the matrix $\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix}$

Rearranging the order of the rows gets us the following row-echelon form.

$$\begin{bmatrix} -2 & 8 & 5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

I circled the pivot positions, as you can see, there is a pivot position in every row. So, by theorem 4, these do span $\mathbb{R}^3$.

Exercise 2. Do the following vectors span $\mathbb{R}^4$?

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Again, we use theorem 4. We put the matrix into a row-echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+R_3 \to R_3}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3+R_4 \to R_4} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last row does not have a pivot position, so by Thm. 4, these do not span $\mathbb{R}^4$.

Exercise 3. Determine if the following systems have a nontrivial solution:

- $x_1 - 3x_2 + 7x_3 = 0, \quad -2x_1 + x_2 - 4x_3 = 0 \quad x_1 + 2x_2 + 9x_3 = 0$
- $-5x_1 + 7x_2 + 9x_3 = 0, \quad x_1 - 2x_2 + 6x_3 = 0$

$x_1 - 3x_2 + 7x_3 = 0$
$-2x_1 + x_2 - 4x_3 = 0$
$x_1 + 2x_2 + 9x_3 = 0$

The augmented matrix looks like:

$$\begin{bmatrix} 1 & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{bmatrix} \xrightarrow[-R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 5 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 8 & 0 \end{bmatrix}$$

Every column has a pivot position, so there are no free variables. Thus, there is no nontrivial solution.

$-5x_1 + 7x_2 + 9x_3 = 0$
$x_1 - 2x_2 + 6x_3 = 0$

The augmented matrix looks like:

$$\begin{bmatrix} -5 & 7 & 9 & 0 \\ 1 & -2 & 6 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 6 & 0 \\ -5 & 7 & 9 & 0 \end{bmatrix} \xrightarrow{5R_1+R_2 \to R_2}$$

$$\begin{bmatrix} 1 & -2 & 6 & 0 \\ 0 & -3 & 39 & 0 \end{bmatrix}$$

We can see that x_3 is a free variable, since there is no pivot position in the third column. So there do exist nontrivial solutions.

Exercise 4. Describe all solutions of $A\mathbf{x} = \mathbf{0}$, for the following matrices. Express your answers in parametric vector form.

$$A = \begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$$

We row-reduce the augmented matrix:

$$\begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

x_2, x_3, x_4 are free variables.

Writing x_1 in terms of these, we get

$x_1 = -3x_2 + 4x_4$.

In parametric vector form, we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$$

We row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \xrightarrow{2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix}$$

x_3, x_4 are free variables

Writing x_1, x_2 in terms of these, we get:

$x_1 = 5x_3 + 7x_4$
$x_2 = -2x_3 + 6x_4$.

In parametric vector form, we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 5. Describe the solutions of the system given by the following augmented matrix. Express your answer in parametric vector form.

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix}$$

We first row-reduce the matrix:

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -7 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \xrightarrow{4R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{bmatrix} \xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2}$$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-3R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in reduced row echelon form.

I circled the pivot positions, we see that x_3 is a free variable.

We can write the solutions as

$x_1 = 5x_3 + 2$

$x_2 = -2x_3 - 1.$

$x_3 = x_3$

In parametric vector form, this is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

Exercise 6. Let A be a 3×2 matrix with 2 pivot positions. Does $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Does $A\mathbf{x} = \mathbf{b}$ have at least 1 solution for every $\mathbf{b}$ in $\mathbb{R}^3$?

First, we note that A must have a row-echelon form that looks like $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$ and a reduced row echelon form that looks like $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

a) The augmented matrix of $A\vec{x} = \vec{0}$ looks like $\left[A \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$ which reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which translates to $x_1 = 0$, $x_2 = 0$. So only the trivial solution solves the equation.

We could also use Theorem 4 here! This explanation kind of shows the reasoning behind Thm 4

b). The augmented matrix of $A\vec{x} = \vec{b}$ looks $\left[A \,\middle|\, \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}\right]$ which reduces to $\begin{bmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & c_3 \end{bmatrix}$, where c_1, c_2, c_3 are whatever you get by applying row operations to b_1, b_2, b_3. Since we can choose b_1, b_2, b_3 to be anything, c_1, c_2, c_3 can be anything. ~~If~~ $c_3 \neq 0$, the system is inconsistent. So $Ax = b$ does not always have a solution.