1. Are the following functions odd, even, or neither??

(a). $f(x) = \sin^2(x)$.

(b) $f(x) = \sin(x+1)$

(c) $x^{1/3}\cos(x^2)$.

(a). This is a product of two odd functions, so it is even

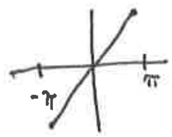
(b). Neither! for example, $f(-1) = \sin(0) = 0$, while $f(1) = \sin(2) \neq \pm 0$

since $f(x) \neq \pm f(-x)$, this is neither even nor odd.

(c). This is a product of an odd function $(x^{1/3})$ and an even $(\cos(x^2))$, so it is odd.

2. Compute the fourier series of the given function on the specified interval:

$$f(x) = x, \quad -\pi < x < \pi.$$



The formula for the fourier series is: $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$.

All we have to do is find the coefficients.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0 \quad \text{since } x \text{ is odd.}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{\pi}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x(\cos nx)\,dx = 0 \quad \text{since } x(\cos(nx)) \text{ is odd.}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi x}{\pi}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx \quad \text{since } x\sin(nx) \text{ is even.}$$

We can now proceed by integration by parts!

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx. \quad = \frac{2}{\pi}\left[-\frac{x}{n}\cos nx\Big|_0^{\pi} - \int_0^{\pi}\left(-\frac{1}{n}\cos(nx)\right)dx\right]$$

$u = x \qquad dv = \sin(nx)\,dx$

$du = dx \qquad v = -\frac{1}{n}\cos nx.$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}\overbrace{\cos(\pi n)}^{(-1)^n} + \frac{\sin(nx)}{n^2}\Big|_0^{\pi}\right] = \frac{2}{\pi}\frac{(-1)^{n+1}\pi}{n} = \frac{(-1)^{n+1}2}{n}$$

Putting everything together, we get:

$$f \sim \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2}{n}\sin(nx).$$

3. The norm of a function, $\|f\| = \sqrt{(f,f)}$ is like the length of a vector in $\mathbb{R}^n$. In particular, show that this norm satisfies the following properties associated with length.

Note: Here I will assume f is continuous (otherwise (a) isn't nec. true)

(a). $\|f\| \geq 0$, and $\|f\| = 0$ if and only if $f = 0$ ← we don't know enough analysis to do this rigorously so I am skipping it.

(b) $\|cf\| = |c|\|f\|$, when c is any real number.

(c) $\|f+g\| \leq \|f\| + \|g\|$.

(a). $\|f\| = \sqrt{\int_{-L}^{L} f^2(x)\,dx}$. First of all, $\int_{-L}^{L} f^2(x) \geq 0$ since f^2 is positive. Thus the square root gives us a real, nonnegative number.

(b) $\|cf\| = \sqrt{\int_{-L}^{L} c^2 f^2(x)\,dx} = \sqrt{c^2 \int_{-L}^{L} f^2(x)\,dx} = |c| \sqrt{\int_{-L}^{L} f^2(x)\,dx} = |c|\,\|f\|$.

(c). We first show that $\|f+g\|^2 \leq (\|f\| + \|g\|)^2$.

$$\|f+g\|^2 = \int_{-L}^{L} (f+g)^2(x)\,dx = \int_{-L}^{L} f^2 + 2fg + g^2\,dx = \int_{-L}^{L} f^2 + g^2\,dx + 2\int_{-L}^{L} fg\,dx$$

$$(\|f\| + \|g\|)^2 = \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| = \int_{-L}^{L} f^2 + g^2\,dx + 2\sqrt{\int_{-L}^{L} f^2\,dx}\sqrt{\int_{-L}^{L} g^2\,dx}$$

$$= \int_{-L}^{L} (f^2 + g^2)\,dx + 2\sqrt{\int_{-L}^{L} f^2\,dx \int_{-L}^{L} g^2\,dx}$$

Note that by cauchy shwarz inequality, ← we mentiond this briefly a long time ago.

$$\int_{-L}^{L} fg\,dx \leq \sqrt{\int_{-L}^{L} f^2\,dx \int_{-L}^{L} g^2\,dx}$$

So $\|f+g\|^2 \leq (\|f\| + \|g\|)^2$. Taking the square root preserves the inequality, so we conclude

$$\|f+g\| \leq \|f\| + \|g\|.$$