Worksheet 26

Sections 306 and 310
MATH 54

Nov 27, 2018

Exercise 1. Find a general solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ for the given matrix A. You can use the fact that the eigenvalues of A are 2, $2+i$, and $2-i$.

$$A = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}$$

I just gave one choice of eigenvector for each eigenvalue.

I used Wolfram Alpha to find that the eigenvalues/vectors are:

$\lambda_1 = 2, \quad v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ Using section 9.5, This corresponds to a solution of

$$e^{2t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Note: I will ask John Lott if he expects a by-hand computation. If so, I will post an example.

The remaining two eigenvalues are $2 \pm i$ with cor. eigenvectors $\begin{bmatrix} 5 \\ -2 \pm i \\ 5 \end{bmatrix}$.

Recall that a formula for 2 lin. ind solutions is:

$$e^{\alpha t}\cos\beta t\,\vec{a} - e^{\alpha t}\sin\beta t\,\vec{b}$$

and

$$e^{\alpha t}\sin\beta t\,\vec{a} + e^{\alpha t}\cos\beta t\,\vec{b}$$

Here, $\alpha = 2$, $\beta = 1$, $\vec{a} = \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ So our two solutions are

$$e^{2t}\cos t\begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix} - e^{2t}\sin t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e^{2t}\begin{bmatrix} 5\cos t \\ -2\cos t - \sin t \\ 5\cos t \end{bmatrix}$$

and

$$e^{2t}\sin t\begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix} + e^{2t}\cos t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e^{2t}\begin{bmatrix} 5\sin t \\ -2\sin t + \cos t \\ 5\sin t \end{bmatrix}.$$

Putting everything together, our general solution is:

$$x(t) = c_1 e^{2t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 5\cos t \\ -2\cos t - \sin t \\ 5\cos t \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 5\sin t \\ -2\sin t + \cos t \\ 5\sin t \end{bmatrix}.$$

2(cont) Our normal frequencies are $\sqrt{3}$, 1. (since our eigenvals are of the form $\pm i$, $\pm\sqrt{3}i$).

The general solution is $c_1\vec{Y}_1 + c_2\vec{Y}_2 + c_3\vec{Y}_3 + c_4\vec{Y}_4$ (Note I am using $y_1, \vec{Y}_1$ to mean different things)

If you solve the initial value problem, you should get $c_4 = 1$, $c_2 = -\sqrt{3}$, $c_1 = c_3 = 0$

Exercise 2. For the coupled mass-spring system governed by (10) on page 536 of the book, assume $m_1 = m_2 = 1$kg, $k_1 = k_2 = k_3 = 1$ N/m and assume initially that $x_1(0) = 0$ m, $x_1'(0) = 0$m/s, $x_2(0) = 2$m and $x_2'(0) = 0$ m/s. Find the normal frequencies and solve the initial value problem.

The book states that the coupled mass system is governed by the following:

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' = -k_2(x_2 - x_1) - k_3 x_2 \end{cases}$$

Substituting our values for m_1, m_2, k_1, k_2, k_3,

$$\begin{cases} x_1'' = -x_1 + x_2 - x_1 = x_2 - 2x_1 \\ x_2'' = -(x_2 - x_1) - x_2 = x_1 - 2x_2 \end{cases}$$

We wish to turn this into a system of first order eqs. Let $y_1 = x_1, y_2 = x_1', y_3 = x_2, y_4 = x_2'$.

So $$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 - 2y_1 \\ y_4 \\ y_1 - 2y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

The matrix has eigenvals and eigenvecs:

$$\lambda = \pm\sqrt{3}\,i \quad \vec{v} = \begin{bmatrix} \pm i/\sqrt{3} \\ -1 \\ \mp i/\sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \\ 0 \end{bmatrix}$$

This corresponds to solutions of the form

$$\vec{Y}_1 = \begin{bmatrix} -\sin(\sqrt{3}t)/\sqrt{3} \\ -\cos(\sqrt{3}t) \\ \sin(\sqrt{3}t)/\sqrt{3} \\ \cos(\sqrt{3}t) \end{bmatrix}, \quad \vec{Y}_2 = \begin{bmatrix} \cos(\sqrt{3}t)/\sqrt{3} \\ -\sin(\sqrt{3}t) \\ -\cos(\sqrt{3}t)/\sqrt{3} \\ \sin(\sqrt{3}t) \end{bmatrix}$$

$$\lambda = \pm i, \quad \vec{v} = \begin{bmatrix} \pm i \\ 1 \\ \pm i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

corresponding solutions are.

$$\vec{Y}_3 = \begin{bmatrix} -\sin t \\ \cos t \\ -\sin t \\ \cos t \end{bmatrix} \quad \vec{Y}_4 = \begin{bmatrix} \cos t \\ \sin t \\ \cos t \\ \sin t \end{bmatrix}$$

Continued in top margin.

Exercise 3. Determine whether the given vector functions are linearly independent or linearly dependent on $(-\infty, \infty)$.

Note: For the solution of this problem I assumed $a \neq 0$.

(a) Show that the Cauchy-Euler equation $at^2y'' + bty' + cy = 0$ can be written as system

$$t\mathbf{x}' = A\mathbf{x}$$

with a constant coefficient matrix A, by setting $x_1 = y/t$ and $x_2 = y'$.

(b) Show that for $t > 0$ any system of the form above has nontrivial solutions of the form $\mathbf{x}(t) = t^r\mathbf{u}$ if and only if r is an eigenvalue of A and $\mathbf{u}$ is a corresponding eigenvector.

(a) Let $x_1 = y/t$, $x_2 = y'$.

We wish to express tx_1', tx_2' in terms of x_1, x_2. Note that

$$tx_1' = t\left(\frac{ty' - y}{t^2}\right) = \frac{ty' - y}{t} = y' - \frac{y}{t} = x_2 - x_1$$

$$tx_2' = ty'' = -\frac{b}{a}y' - \frac{c}{a}\frac{y}{t} = -\frac{b}{a}x_2 - \frac{c}{a}x_1$$

$$t\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = x_1\begin{bmatrix} -1 \\ -c/a \end{bmatrix} + x_2\begin{bmatrix} 1 \\ -b/a \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -c/a & -b/a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

as desired.

(b) Since this is an if and only if statement, we have to prove two directions.

$\Rightarrow$ We first assume that $\vec{x}(t) = t^r\vec{u}$ is a solution.

then, $t\vec{x}' = A\vec{x}$. Substituting in our expression for $\vec{x}(t)$, we get:

$t(rt^{r-1}\vec{u}) = At^r\vec{u}$. Since $t \neq 0$, we can divide both sides by t^r, getting $r\vec{u} = A\vec{u}$. So r is an eigenval with corr. eigenvec $\vec{u}$ by definition.

$\Leftarrow$ We now assume r is an eigenval of A with corr. eigenval $\vec{u}$ and show that $t^r\vec{u}$ is a solution to the system. Let $\vec{x} = t^r\vec{u}$. Then:

$$t\vec{x}' = t(rt^{r-1}\vec{u}) = t^r(r\vec{u}) = t^r(A\vec{u}) = A$$

by def of eigenval/vec.

So we see that $t^r\vec{u}$ is indeed a solution to $t\vec{x}' = A\vec{x}$