Worksheet 25

Sections 306 and 310

MATH 54

Nov 25, 2018

Exercise 1. Consider $\mathbf{x}'(t) = A\mathbf{x}(t)$, $t \geq 0$, where $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

(a) Show that the eigenvalues are -1,-3 and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are the corresponding eigenvectors.

(b) Sketch the tractory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u}_1$.

(c) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = -\mathbf{u}_2$.

(d) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u}_1 - \mathbf{u}_2$.

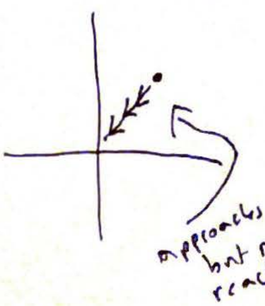
(a). $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is indeed an eigenvec with corr. eigenval -1. ✓

$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is indeed an eigenvec with corr eigenval -3.

(b). The solution corr to this IVP is

$\vec{x}(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

As t gets bigger, $\vec{x}(t)$ gets closer to 0. every point on the trajectory is a scalar mult of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



approaches 0 but never reaches it.

(c) Solution is $\vec{x}(t) = -e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(d). Solution is $\vec{x}(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-3t} \\ e^{-t} + e^{-3t} \end{bmatrix}$.

Facts about this:

- Approaches origen as $t \to \infty$.
- both coords are always positive.
- First coord is always less than second. so above line $y = x$.

Exercise 2. Give a general solution to

$$\mathbf{x}'(t) \quad \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$$

Eigenvalues are solutions to: $(6-\lambda)(1-\lambda)+6 = \lambda^2 - 7\lambda + 12 = (\lambda-3)(\lambda-4) = 0$
So $\lambda = 3, 4$.

We now find eigenvecs:

$\lambda = 3$: $\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right]$ $-x_1 + x_2 = 0$
$x_1 = x_2$.

So eigenvecs are of form $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 4$ $\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right]$. So $x_1 = 3x_2$.

So eigenvecs are of form

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Putting everything together, our general solution is:

$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Exercise 3. Use the substitutions $x_1 = y$, $x_2 = y'$ to convert $ay'' + by' + cy = 0$ into a normal system. Show that the characteristic equation of the system is the same as the aux equation for the original equation.

We are assuming $a \neq 0$.

Let $x_1 = y$, $x_2 = y'$. Then:

$\vec{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{b}{a}x_2 - \frac{c}{a}x_1 \end{bmatrix}$

$= x_1 \begin{bmatrix} 0 \\ -c/a \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -b/a \end{bmatrix} =$

$\begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix} \vec{x}$. So

our normal system is $\vec{x}' = A\vec{x}$ where

$A = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}$. The char equation

of the system is the same as the char equation of the matrix:

$(-\lambda)(-b/a - \lambda) + c/a = 0$

This rearranges to

$b/a\lambda + \lambda^2 + c/a = \lambda^2 + b/a\,\lambda + c/a = 0$.

Multiplying both sides by a, we get:

$a\lambda^2 + b\lambda + c = 0$

which is indeed the aux equation of the equation

$ay'' + by' + cy = 0$