Worksheet 23

Sections 306 and 310
MATH 54

Nov 8, 2018

Exercise 1. Use variation of parameters to find a general solution for the following:

$$y'' + y = \sec(t)$$

The first step is to find the general solution of the homogenous equation.

The aux equation is $r^2+1=0 \Rightarrow r = \pm i$. So the homogenous equations have solutions $y(t) = c_1 \sin t + c_2 \cos t$. For variation of parameters, we write our particular solution in the fol. form:

$$y_p(t) = v_1(t)\sin t + v_2(t)\cos t.$$

For some functions v_1, v_2. Recall from the book that v_1', v_2' have to satisfy the following system:

$$\begin{cases} v_1' \sin t + v_2' \cos t = 0 \\ v_1' \cos t - v_2' \sin t = \sec t. \end{cases}$$

From the first equation, we get $v_1' = -\frac{\cos t}{\sin t} v_2'$. Substituting this into the second equation, we get. $-\frac{\cos^2 t}{\sin t} v_2' - v_2' \sin t = \sec t \Rightarrow -\left(\frac{\cos^2 t + \sin^2 t}{\sin t} v_2'\right) = \frac{1}{\cos t} \Rightarrow v_2' = -\tan t.$

$v_1' = -\frac{\cos t}{\sin t} v_2' = -\frac{\cos t}{\sin t}(-\tan t) = 1$. Now we integrate to find v_1, v_2.

$v_1(t) = \int v_1' dt = \int 1 dt = t + c_1$

$v_2(t) = \int v_2' dt = \int -\tan t \, dt = \int \frac{-\sin t}{\cos t} dt = \int \frac{du}{u} = \ln|u| + c_2 = -\ln|\cos t| + c_2$

$u = \cos t$
$du = -\sin t \, dt$

Note: since we just need one particular solution, we can let $c_1 = c_2 = 0$.

So $y_p = t\sin t + \cos t \ln|\cos t|$.

Putting everything together, we get the following general solution:

$$y(t) = c_1 \sin t + c_2 \cos t + t \sin t + \cos t \ln|\cos t|.$$

Exercise 2. Use a combination of the law of superposition, undetermined coefficients, and variation of parameters to solve the following:

$$y'' + y = 3\sec(t) - t^2 + 1$$

Note: You already did the variation of parameters part in exercise 1!
Why might someone want to use this method rather than just use only variation of parameters?

To get a particular solution to this dif. eq, we add the particular solutions of the following two equations:

(1) $y''+y=3\sec(t)$

(2) $y''+y=1-t^2$.

This follows from the law of superposition.

Note that we know the particular solution to $y''+y=\sec t$ from exercise one. By the law of sup, we can multiply this by 3 to get a solution of $y''+y=3\sec t$.

So a particular solution for (1) is $3(t\sin t+\cos t\ln|\cos t|)$

We now use undetermined coefficients to find a solution for $y''+y=1-t^2$.

Our guess is of the form $y_p = (A+Bt+Ct^2)t^s$. but $s=0$ since 0 (coming from e^{0t}) is not a root of the aux equation.

So $y_p = A+Bt+Ct^2$.

$y_p' = B+2Ct$

$y_p'' = 2C$

Plugging this into equation (2), we get

$2C + A + Bt + Ct^2 = 1-t^2$.

So

$A+2C=1$, $B=0$, $C=-1$ $\Rightarrow$ $A=3$, $B=0$, $C=-1$.

So $y_p = 3-t^2$.

Recall from exercise 1 that the general solution to the homogeneous equation $y''+y=0$ is $c_1\sin t + c_2\cos t$.

Putting everything together, we get that the general solution is

$y(t) = c_1\sin t + c_2\cos t + 3t\sin t + 3\cos t\ln|\cos t| - t^2 + 3$.

Exercise 3. Use the method of variation of parameters to show that:

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + \int_0^t f(s) \sin(t-s) ds$$

is a general solution to the differential equation $y'' + y = f(t)$ where $f(t)$ is a continuous function. Hint: Use a trig identity for $\sin(t-s)$.

As always, the first step is finding the gen solution to the homogenous equation. Luckily for us, in previous problems we established that $y''+y=0$ has solutions $c_1 \sin t + c_2 \cos t$. For variation of parameters, we write our particular solution in the following form:

$$y_p(t) = v_1(t) \sin t + v_2(t) \cos t. \quad \text{for some functions } v_1, v_2.$$

$v_1'(t), v_2'(t)$ must satisfy the following system:

$$\begin{cases} v_1' \sin t + v_2' \cos t = 0 \\ v_1' \cos t - v_2' \sin t = f(t). \end{cases}$$

Rearranging the first equation, we get $v_1' = -\frac{\cos t}{\sin t} v_2'$. Substituting this into the second equation, we get

$$\frac{-\cos^2 t}{\sin t} v_2' - v_2' \sin t = f(t) \Rightarrow v_2' \left(\frac{-(\cos^2 t + \sin^2 t)}{\sin t} \right) = f(t) \Rightarrow v_2' = -\sin t \, f(t)$$

$$v_1' = \cos t \, f(t).$$

So one set of choices for v_1, v_2 are $v_1 = \int_0^t \cos s \, f(s) \, ds$, $v_2 = \int_0^t -\sin s \, f s \, ds$.

Putting everything together, we get that

$$y_p = \sin t \int_0^t \cos(s) f(s) \, ds - \cos t \int_0^t \sin(s) f(s) \, ds = \int_0^t (\sin t \cos s - \cos t \sin s) f(s) \, ds =$$

$\int_0^t f(s) \sin(t-s) \, ds$ using the sum/difference of angles trig identity.

So the general solution is

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + \int_0^t f(s) \sin(t-s) ds$$

as desired.