Worksheet 21

Sections 306 and 310
MATH 54

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Exercise 1. Find a general solution to the given differential equations:

(a) $y'' + y = 0$

(b) $y'' - 10y' + 26 = 0$

(c) $y'' - 4y' + 7y = 0$

26
y4

(a). The auxiliary equation is $r^2+1=0$, which has solutions $\pm i$
So the general solution is $y(t) = c_1 e^{0t}\sin t + c_2 e^{0t}\cos t =$
$c_1 \sin t + c_2 \cos t$.

(b). The auxiliary equation is $r^2 - 10r + 26 = 0$. Using quadratic formula,
we get $r = \dfrac{10 \pm \sqrt{100-104}}{2} = 5 \pm i$
so $y(t) = c_1 e^{5t}\sin t + c_2 e^{5t}\cos t$.

(c) The auxiliary equation is $r^2 - 4r + 7 = 0$ Using quad formula,
we get $r = \dfrac{4 \pm \sqrt{-12}}{2} = 2 \pm \sqrt{3}\, i$
So $y(x) = c_1 e^{2x}\sin(\sqrt{3}x) + c_2 e^{2x}\cos(\sqrt{3}\, x)$

Exercise 2. To see the effect of changing the parameter b in the initial value problem

$y'' + by' + 4y = 0$; $y(0) = 1$; $y'(0) = 0$

Solve the problem for $b = 5$, 4, and 2 and sketch the solutions.

$\underline{b=5}$ Aux equation is $r^2+5r+4 = (r+4)(r+1)=0$.

So $y(t) = c_1 e^{-4t} + c_2 e^{-t}$

$y'(t) = -4c_1 e^{-4t} - c_2 e^{-t}$.

We now solve the initial value problem:

$y(0) = c_1 + c_2 = 1$

$y'(0) = -4c_1 - c_2 = 0$

so $c_1 = -\frac{1}{3}$, $c_2 = \frac{4}{3}$.

So $y(t) = -\frac{1}{3}e^{-4t} + \frac{4}{3}e^{-t}$.

$\underline{b=4}$ Aux equation is $r^2+4r+4 = (r+2)^2 = 0$. so $r = \pm 2$.

So $y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

$y'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$

We now solve the initial value problem:

$y(0) = c_1 + c_2 = 1$

$y'(0) = -2c_1 + c_2 = 0$

So $c_1 = \frac{1}{3}$, $c_2 = \frac{2}{3}$

So $y(t) = \frac{1}{3}e^{-2t} + \frac{2}{3}te^{-2t}$.

$\underline{b=2}$ Aux equation is $r^2+2r+4=0$ Using quad formula, $r = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i$.

So $y(t) = c_1 e^{-t}\sin(\sqrt{3}t) + c_2 e^{-t}\cos(\sqrt{3}t)$.

$y'(t) = e^{-x}\left(c_1\left(\sqrt{3}\cos(\sqrt{3}x) - \sin(\sqrt{3}x)\right) - c_2\left(\sqrt{3}\sin(\sqrt{3}x) + \cos(\sqrt{3}x)\right)\right)$

We now solve the initial value problem:

$y(0) = c_2 = 1$

$y'(0) = \sqrt{3}c_1 - c_2 = 0$ so $c_1 = \frac{1}{\sqrt{3}}$, $c_2 = 1$

$y(t) = \frac{1}{\sqrt{3}}e^{-x}\sin(\sqrt{3}t) + e^{-x}\cos(\sqrt{3}t)$

Exercise 3. Find a general so'ution to the following higher-order equation:

$$y''' - y'' + y' + 3y = 0$$

The auxiliary equation is $r^3 - r^2 + r + 3 = 0$

From using guess and check, we see that $r = -1$. So $(r+1)$ is a factor. You can use polynomial long devision to finish factoring!

$(r+1)(r^2 - 2r + 3) = 0$. So the roots are $r = -1$, $1 \pm \sqrt{2}i$

so $y(t) = c_1 e^{-t} + c_2 e^t \sin(\sqrt{2}t) + c_3 e^t \cos(\sqrt{2}t)$.

Exercise 4. Prove the sum of angles formula for the sine function by following these steps. Let x be a fixed constant.

(a) Let $f(t) = \sin(x + t)$. Show that $f''(t) + f(t) = 0$, $f(0) = \sin x$, and $f'(0) = \cos(x)$.

(b) Use the auxiliary techniqu to solve the initial value problem $y'' + y = 0$, $y(0) = \sin(x)$, and $y'(0) = \cos(x)$.

(c) By uniqueness, the solut'on in part (b) is the same as $f(t)$ from part (a). Write this equality, this should be the standard sum of angles formula for sin(x+t).

(a). Before verifying these things, we compute the first three derivatve, with respect to t (ie we treat x as a constant)

$f'(t) = \cos(x+t)$, $f''(t) = -\sin(x+t)$.

So indeed, $f''(t) + f(t) = -\sin(x+t) + \sin(x+t) = 0$,

$f(0) = \sin(x+0) = \sin(x)$. as desired.

$f'(0) = \cos(x+0) = \cos(x)$

(b). The aux eqnation is $r^2 + 1 = 0 \Rightarrow r = \pm i$.

So $y(t) = c_1 \sin t + c_2 \cos t$, $y'(t) = c_1 \cos t - c_2 \sin t$.

We now solve initial value problem:

$y(0) = c_2 \cos 0 = c_2 = \sin x$.

$y'(0) = c_1 \cos 0 = c_1 = \cos(x)$.

So $c_1 = \cos x$, $c_2 = \sin x$

Putting this together, $y(t) = \cos x \sin t + \sin x \cos t$.

This is the sum of angles formula for sine.

(c). Setting $y(t) = f(t)$, we get $\cos x \sin t + \sin x \cos t = \sin(x+t)$