# Worksheet 2

Sections 306 and 310
MATH 54

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All 3 have infinitely many solutions!!

**Exercise 1.** Find the general solution of the systems whose augmented matrices are shown below. Some of these matrices may look familiar :).

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\begin{bmatrix} 1 & 4 & 0 & 1 \\ 2 & 7 & 0 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 0 & 1 \\ 2 & 7 & 0 & 10 \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & -1 & 0 & 8 \end{bmatrix} \xrightarrow{4R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 33 \\ 0 & -1 & 0 & 8 \end{bmatrix} \xrightarrow{-R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 & 33 \\ 0 & 1 & 0 & -8 \end{bmatrix}$$

So $x = 33$, $y = -8$, $z$ is a free variable

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix} \xrightarrow{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -7 \end{bmatrix} \xrightarrow{R_2+R_1 \to R_1} \begin{bmatrix} 1 & -2 & 0 & -4 \\ 0 & 0 & 1 & -7 \end{bmatrix}$$

$z = -7$
$x = 2y - 4$.
$y$ is free.

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \xrightarrow{R_1+R_3 \to R_3}$$

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix} \xrightarrow{-4R_2+R_3 \to R_3} \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 7x_2 - 6x_4 + 5$
$x_3 = 2x_4 - 3$

$x_2, x_4$ are free.

**Exercise 2.** Find $h, k$ such that the system below has: (a) no solutions, (b) a unique solution, and (c) infinitely many solutions.

$$x + hy = 2$$

$$4x + 8y = k$$

The augmented matrix is:

$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix}$. One possible row echelon form is $\begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-2 \end{bmatrix}$.

(a) This is inconsistent if and only if the last row is of the form

$[0, 0, a]$, $a \neq 0$.

So we want $8-4h = 0$ and $k-2 \neq 0$

So

$h=2$, $k \neq 8$.

(b) There is a unique solution if and only if the row echelon form looks like:

$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \end{bmatrix}$, so

$8-4h \neq 0$. So

$h \neq 2$.

(c) The only other possibility is $8-4h = 0$, $k-2 = 0$

which simplifies to

$h=2$, $k=8$.

**Exercise 3.** Solve the following systems:

- $x_1 - 3x_2 + 4x_3 = -4, \quad 3x_1 - 7x_2 + 7x_3 = -8, \quad -4x_1 + 6x_2 - x_3 = 7$
- $x_1 - 3x_2 = 5, \quad -x_1 + x_2 + 5x_3 = 2, \quad x_2 + x_3 = 0$

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix} \xrightarrow[4R_1+R_3 \to R_2]{-3R_1+R_2\to R_2} \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix} \xrightarrow{-3R_2+R_3\to R_3} \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & -21 \end{bmatrix}$$

inconsistent!

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_1+R_2\to R_2}$$

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{bmatrix} \xrightarrow{2R_2+R_3\to R_3}$$

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{bmatrix} \xrightarrow{\frac{1}{7}R_3\to R_3} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_3+R_2\to R_2}$$

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{3R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$x_1 = 2, \quad x_2 = -1, \quad x_3 = 1.$

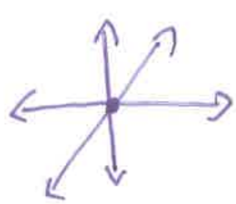
**Exercise 4.** A system of linear equations with more equations than unknowns is sometimes called *overdetermined.* Can such a system be consistent? Illustrate your answer with a specific system of 3 equations and 2 unknowns. (It may be helpful to draw a picture in the plane!)

Yes! Any system of 3 lines that intersects at one point works!

For example:

$$\begin{aligned} x &= 0 \\ y &= 0 \\ x &= y \end{aligned}$$

is consistent, $x=0, y=0$ is a solution.



**Exercise 5.** Write the following vector equation as a matrix equation and also as a system of linear equations. Solve the system.

$$x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Matrix Equation: $\begin{bmatrix} -2 & 8 & 1 \\ 3 & 5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

linear system: $-2x_1 + 8x_2 + x_3 = 0$
$3x_1 + 5x_2 - 6x_3 = 0$

To solve, we row-reduce the augmented matrix.

$$\begin{bmatrix} -2 & 8 & 1 & 0 \\ 3 & 5 & -6 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & -4 & -\frac{1}{2} & 0 \\ 3 & 5 & -6 & 0 \end{bmatrix} \xrightarrow{-3R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -4 & -\frac{1}{2} & 0 \\ 0 & 17 & -\frac{9}{2} & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{17}R_2 \to R_2} \begin{bmatrix} 1 & -4 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{9}{34} & 0 \end{bmatrix} \xrightarrow{-4R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & -\frac{53}{34} & 0 \\ 0 & 1 & -\frac{9}{34} & 0 \end{bmatrix}.$$

So the solutions satisfy $x - \frac{53}{34} z = 0$, $y - \frac{9}{34} z = 0$

**Exercise 6.** Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ 8 \end{bmatrix}$$

b is a linear combination of $a_1, a_2, a_3$ if and only if there exist $x_1, x_2, x_3$ such that:

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ 8 \end{bmatrix}.$$

The augmented matrix of this vector equation is

$$\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & 8 \end{bmatrix}$$ which reduces to $$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is inconsistent.
So b is not a linear combination of $a_1, a_2, a_3$.

**Exercise 7.** Write the following products as linear combinations of the columns of the matrix. Use this to compute the product.

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} =$$

$$2\begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} - 3\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} =$$

$$\begin{bmatrix} 12 \\ -8 \\ 14 \end{bmatrix} + \begin{bmatrix} -15 \\ 9 \\ -18 \end{bmatrix} =$$

$$\begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

**Exercise 8.** Write the augmented matrix for the linear system that corresponds to the matrix equation $Ax = \mathbf{b}$. Then solve the system and write the solution as a vector.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & 2 \\ 0 & 5 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & 1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \xrightarrow{3R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 7 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \xrightarrow{\frac{1}{7}R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{5}{7} & \frac{1}{7} \\ 0 & 5 & 3 & -1 \end{bmatrix} \xrightarrow{-5R_2 + R_3 \to R_3}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{5}{7} & \frac{1}{7} \\ 0 & 0 & -\frac{4}{7} & -\frac{12}{7} \end{bmatrix} \xrightarrow{-\frac{7}{4}R_3 \to R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{5}{7} & \frac{1}{7} \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[-R_3 + R_1 \to R_1]{-\frac{5}{7}R_3 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{-R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$