Worksheet 18

Sections 306 and 310
MATH 54

October 23, 2018

Exercise 1. Consider $\mathbb{P}_2$ with the following inner product:

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

Let $p(t) = 3t - t^2$ and $q(t) = 3 + 2t^2$. Compute the following:

(a) $\langle p, q \rangle$

(b) $||p||$, $||q||$

(c) $\text{proj}_W q$, where $W = \text{span}\{p\}$.

(a). $\langle p,q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) = (-4)(5) + (0)(3) + (2)(5) = -10$

(b) $\|p\| = \sqrt{\langle p,p \rangle} = \sqrt{(p(-1))^2 + (p(0))^2 + (p(1))^2} = \sqrt{(-4)^2 + 0^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$.

$\|q\| = \sqrt{\langle q,q \rangle} = \sqrt{5^2 + 3^2 + 5^2} = \sqrt{59}$.

(c) $\text{proj}_W q = \dfrac{\langle p,q \rangle}{\langle p,p \rangle} p = \dfrac{-10}{20}(3t - t^2) = -\dfrac{3t - t^2}{2}$.

Exercise 2. Use the axioms of inner product spaces to prove the following: For any vector $\mathbf{v}$ in an inner product space V,

$$\langle \mathbf{v}, \mathbf{0} \rangle = \langle \mathbf{0}, \mathbf{v} \rangle = 0$$

By axiom 1 (page 378), $\langle \vec{v}, \vec{0} \rangle = \langle \vec{0}, \vec{v} \rangle$.

By axiom 3, $\langle \vec{0}, \vec{v} \rangle = \langle 0 \cdot \vec{u}, \vec{v} \rangle = 0 \langle \vec{u}, \vec{v} \rangle = 0$

Putting everything together, we get

$\langle \vec{v}, \vec{0} \rangle = \langle \vec{0}, \vec{v} \rangle = 0$, as desired.

Exercise 3. True of false! Justify!

(a) There are symmetric matrices that are not orthogonally diagonalizable. F

(b) An orthogonal matrix is always orthogonally diagonalizeable. F

(c) The dimenaion of the eigenspace is sometimes less than the multiplicity of the corresponding eigenvalue.

(d) The dimension of an eigenspace of a symmetrix matrix is sometimes less than the multiplicity of the corresponding eigenvalue.

(a). False, Thm 2 says that a matrix is orth. diag-able if and only if it is a symmetric matrix. So every sym matrix has to be orthogonally diagonalizable.

(b) False. There exist nonsymmetric orth. matrices, for example $\frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$
By thm 2, these matrices cannot be orth. diagonalizable, since they are not symmetric.

(c). True! You can verify that the eigenspace corresponding to $\lambda = 1$ for $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ has dimension of only 1, even though the eigenvalue has multiplicity 2.

(d). False! See part b of Thm 3 on page 399.

Exercise 4. The following matrix has eigenvalues $\lambda = -2, 7$. Orthongally diagonalize the matrix:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

(note: For this problem I just did a sketch of the procedure! If you have q's about details, email me).

For $\lambda = 7$,

You can compute that a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ Use Gram-Schmidt to turn this into an orthogonal basis: $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \right\}$ and then normalize to turn this into a orthonormal basis. $\left\{ \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix} \right\}$.

For $\lambda = -2$, You can compute that a basis for the eigenspace is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \right\}$ which normalizes to $\begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$.

So $A = PDP^{-1}$ where $P = \begin{bmatrix} -1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ & 5/\sqrt{45} & 2/3 \end{bmatrix}$, $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

Exercise 5. Use the Cauchy-Schwarz inequality to show that $\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}$. HINT: Use the vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Cauchy-Schwarz says: $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|$.

Letting $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and using the dot product ~~[illegible]~~ as our inner product, we get:

$$|a+b| \leq \sqrt{a^2+b^2}\sqrt{2}.$$

Dividing both sides by 2, we get

$$\frac{|a+b|}{2} \leq \frac{\sqrt{a^2+b^2}\sqrt{2}}{2}.$$

Since both sides are positive, we can square both sides without worrying about changing the ~~[illegible]~~ direction of the inequality.

$$\left(\frac{|a+b|}{2}\right)^2 \leq \frac{2(a^2+b^2)}{4}$$

which reduces to

$$\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2},$$

as desired.