Worksheet 17

Sections 306 and 310
MATH 54

October 18, 2018

Exercise 1. Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ by using the normal equations for $\hat{\mathbf{x}}$

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

The normal equations are.

on page 363

$$A^T A \vec{x} = A^T \vec{b}.$$

By thm 13, the solutions to this equation are the least-squares solutions.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}.$$

So it remains to solve

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \vec{x} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}.$$

Using row reduction, one can see that this has solution $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Exercise 2. True and false! Justify your answers! A is $m \times n$ and $\mathbf{b}$ is in $\mathbb{R}^m$.

(a) If $\mathbf{b}$ is in the column space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ is a least-squares solution.

(b) The least-squares solution of $A\mathbf{x} = \mathbf{b}$ is the point in the column space of A closest to $\mathbf{b}$

(c) A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a list of weights that when, applied to the columns of A, produces the orthogonal projection of $\mathbf{b}$ onto Col A.

(a). True! A LSS is a vector $\hat{x}$ such that $\|A\hat{x} - b\|$ is as small as possible. If $\hat{x}$ is a solution to $A\vec{x} = \vec{b}$, then $\|A\hat{x} - \vec{b}\| = \|\vec{b} - \vec{b}\| = \|\vec{0}\| = 0$, which is definitely as small as possible, since the magnitude of a vector is always ≥ 0.

(b). False. A LSS is a vector $\hat{x}$ such that $A\hat{x} = \hat{b}$, where $\hat{b}$ is as close to $\vec{b}$ as possible.

(c) True. Note that the $\hat{b}$ in the previous part is in fact $\text{proj}_{\text{Col} A} \vec{b}$. So $\hat{x}$ is any vector such that

$$A\hat{x} = \text{proj}_{\text{Col} A} \vec{b}.$$

In other words, $\hat{x}$ serves as list of weights such that, when applied to columns of A, give $\text{proj}_{\text{Col} A} \vec{b}$.

Exercise 3. Find the orthogaonal projection of **b** onto Col A and use this to find a least-squares solution of $A\mathbf{x} = \mathbf{b}$

$$A = \begin{array}{c} \begin{array}{cc} \vec{a}_1 & \vec{a}_2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \end{array}, \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

Explain why this method would be more difficult for the matrix given in exercise 1.

We first compute $\hat{b} = \operatorname{proj}_{\operatorname{Col} A} \vec{b} = \frac{\vec{b}\cdot\vec{a}_1}{\vec{a}_1\cdot\vec{a}_1}\vec{a}_1 + \frac{\vec{b}\cdot\vec{a}_2}{\vec{a}_2\cdot\vec{a}_2}\vec{a}_2 =$

$$3\,\vec{a}_1 + \tfrac{1}{2}\vec{a}_2 = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}.$$ ✡

From the discussion in part c of the prev. exercise, we see that $\hat{x}$ is an LSS. if and only if it is a solution to

$$A\vec{x} = \hat{b} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}.$$

We observe that $\begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}$ serve as weights for the column vectors of A that yield $\begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$.

So $\hat{x} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$ is an LSS.

This ~~method works~~ formula marked with ✡ only works since the columns of A are orthogonal. This is not the case in exercise 1.