Worksheet 16

Sections 306 and 310
MATH 54

October 16, 2018

Exercise 1. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in span$\{\mathbf{u}\}$ and one orthogonal to it.

$$\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

We wish to write $\vec{y} = \hat{y} + \vec{z}$, where $\hat{y}$ is in span$\{\vec{u}\}$ and $\vec{z}$ is orthogonal to it. Let W be the subspace spanned by $\vec{u}$.

Then: $\hat{y} = \text{proj}_W \vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{8-9}{16+9}\begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -4/25 \\ 3/25 \end{bmatrix}$

and $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -4/25 \\ 3/25 \end{bmatrix} = \begin{bmatrix} 54/25 \\ 72/25 \end{bmatrix}$

So $\vec{y} = \begin{bmatrix} -4/25 \\ 3/25 \end{bmatrix} + \begin{bmatrix} 54/25 \\ 72/25 \end{bmatrix}$.

From the following picture, we can see that the desired distance is $\|\vec{z}\| = \sqrt{\left(\frac{54}{25}\right)^2 + \left(\frac{72}{25}\right)^2}$.

(Sorry, I accidentally copied the wrong numbers, which made the numbers gross).

Exercise 2. True and false! Justify your answers!

(a) If A is an $n \times n$ matrix with orthogonal columns, then it is invertible.

(b) If a set $\{\mathbf{u_1}, \dots \mathbf{u_p}\}$ has the property that $\mathbf{u_i} \cdot \mathbf{u_j} = 0$ whenever $i \neq j$ then S is an orthnormal set.

(c) If c is not 0, then the orthogaonl projection of $\mathbf{y}$ onto a vector $\mathbf{u}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{u}$.

(a). False! For example $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ has orth. columns, but is not invertible.

The statement "If A is an $n \times n$ matrix with orthogonal, nonzero columns, then it is invertible", is true, however.

(b) False! The $\vec{u}_i$ also have to be unit vectors for the set to be an orthonormal set.

(c) True! You can see it by drawing a picture, or by do

$$\text{proj}_{c\vec{u}} \vec{y} = \frac{\cancel{c}\vec{u} \cdot \vec{y}}{\cancel{c}\vec{u} \cdot \cancel{c}\vec{u}} (\cancel{c}\vec{u}) = \frac{\vec{u} \cdot \vec{y}}{\vec{u} \cdot \vec{u}} \vec{u} = \text{proj}_{\vec{u}} \vec{y}.$$

Warning: You can cancel out scalars in dot product computations, but you can't cancel out vectors.

Exercise 3. Let W be the subspace spanned by the $\mathbf{v}$'s and write $\mathbf{y}$ as a sum of a vector in W and a vector orthogonal to W.

We can verify that these do indeed form an orthogonal basis for W.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

What is the closest point in W to $\mathbf{y}$?

Again, we want to write

$\vec{y} = \hat{y} + \vec{z}$, where $\hat{y}$ is in W and $\vec{z}$ is orthogonal to W.

Using the formula in the orth. decomp. thm,

we get:

$$\hat{y} = \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \frac{\vec{y}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3}\vec{v}_3 =$$

$$\frac{1}{3}\vec{v}_1 + \frac{14}{3}\vec{v}_2 - \frac{5}{3}\vec{v}_3 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

and $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$

$$\text{So } y = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

The closest point in W to $\vec{y}$ is $\hat{y} = \operatorname{proj}_W \vec{y} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$

Exercise 4. Find an orthogonal basis for col(A).

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$

We call the columns of this matrix $\vec{x}_1, \vec{x}_2$, and $\vec{x}_3$ and perform the Gram-Schmidt process on these vectors to build an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \vec{x}_2 - (-3)\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 =$$

$$\vec{x}_3 - \frac{1}{2}\vec{v}_1 - \frac{5}{2}\vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

So an orthogonal basis for W is

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}.$$

This word was missing on original the printout, and is very important

Exercise 5. Without looking at the proof in the book, show that a $\boxed{\text{nonzero}}$ set of orthogonal vectors is linearly independent.

Let $\{\vec{u}_1, \dots \vec{u}_p\}$ be an orthogonal set of nonzero vectors. Suppose there exist $c_1, \dots c_p$ such that $c_1\vec{u}_1 + \dots + c_p\vec{u}_p = 0$. In order to show $\{\vec{u}_1, \dots, \vec{u}_p\}$ is lin ind, we just have to show

$$c_1 = \dots = c_p = 0$$

We first show that $c_1 = 0$. We dot both sides by $\vec{u}_1$:

$$\vec{u}_1 \cdot (c_1 \vec{u}_1 + \dots + c_p\vec{u}_p) = \vec{u}_1 \cdot \vec{0} = 0.$$

$$\Downarrow$$

$$c_1 \vec{u}_1 \cdot \vec{u}_1 + c_2 \vec{u}_1 \cdot \vec{u}_2 + c_3 \vec{u}_1 \cdot \vec{u}_3 + \dots + c_p \vec{u}_1 \cdot \vec{u}_p = 0.$$

Because the vectors are orthogonal, most of these terms are 0. we are left with

$$c_1 \|\vec{u}_1\|^2 = 0 \quad , \text{since } \vec{u}_1 \cdot \vec{u}_1 = \|\vec{u}_1\|^2.$$

Since $\vec{u}_1 \neq \vec{0}$, $\|\vec{u}_1\| \neq 0$. so $c_1 = 0$.

We can use the same reasoning to show that all the c_i's are 0. So $\{\vec{u}_1, \dots, \vec{u}_p\}$ are lin ind by def.