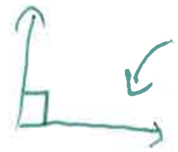
# Worksheet 15

Sections 306 and 310
MATH 54

October 11, 2018

**Exercise 1.** Find a unit vector in the direction of the given vector. Draw a picture of what an orthogonal vector would look like.

$$\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$$



orthogonal vectors form a right angle.

We divide by the magnitude of the vector, which is
$\sqrt{(-6)^2 + 4^2 + (-3)^2} = \sqrt{36+16+9} = \sqrt{61}.$

So a unit vector in the same direction is $\frac{1}{\sqrt{61}}\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

**Exercise 2.** True and false! Justify your answers!

(a) For any scalar $c$, $||c\mathbf{v}|| = c||\mathbf{v}||$.

(b) If $\mathbf{v}$ is orthoganal to every vector in a subspace $W$, then $\mathbf{v}$ is in $W^{\perp}$.

(c) If $||\mathbf{u}||^2 + ||\mathbf{v}||^2 = ||\mathbf{u}+\mathbf{v}||^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(d) For an $m \times n$ matrix $A$, vectors in nul $A$ are orthogonal to vectors in row $A$.

(a) False! When $c$ is negative, $||c\vec{v}||$ is positive, but $c||v||$ is negative.
To fix this, $||c\vec{v}|| = |c|\,||\vec{v}||$ is a true statement.

(b) True! This follows from the definition of being orthogonal to a subspace.

(c) True. See Thm 2 on page 336.

(d) True! See Thm 3. on page 337.

If you are interested in a brief proof, Let $\vec{x}$ be in nul $A$, and
let $A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix}$. Then $A\vec{x} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix}\vec{x} = \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x} \\ \vec{v}_3 \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So $\vec{x}$ is $\perp$ to all the row vectors, and thus everything in row $A$.

These are rows.

Let me know if you have questions :)

**Exercise 3.** Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthoganal basis for $\mathbb{R}^3$. Then express $\mathbf{x}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

We check to make sure every pair is orthogonal.

$\vec{v}_1 \cdot \vec{v}_2 = 6 - 6 + 0 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 2 + 2 - 4 = 0$,

Since we have a set of 3 orth. vectors in $\mathbb{R}^3$, they form an orthogonal basis of $\mathbb{R}^3$.

Since the basis is orthogonal, we can use the formula in Thm 5 (page 341) to find $c_1, c_2, c_3$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{x}$

$$c_1 = \frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{15 + 9 + 0}{9 + 9} = \frac{24}{18} = \frac{4}{3}$$

$$c_2 = \frac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = \frac{10 - 6 - 1}{4 + 4 + 1} = \frac{3}{9} = \frac{1}{3}.$$

$$c_3 = \frac{\vec{x} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} = \frac{5 - 3 + 4}{1 + 1 + 16} = \frac{6}{18} = \frac{1}{3}$$

So $\vec{x} = \frac{4}{3}\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

I always forget these formulas, but the proof on page 341 helps me remember :)

**Exercise 4.** For what values of $b$ is the following matrix diagonalizable?

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

This is diagonalizable if and only if $b = 0$.

Case 1: $b = 0$

Then $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ which is a diagonal matrix, which is diagonalizeable.

Case 2: $b \neq 0$. We know $A$ has only one eigenvalue, $a$, since $A$ is triangular and thus its eigenvalues are on the diagonal entries. We now find the eigenspace by finding the null space of $A - aI = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 0 & b & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Since $b \neq 0$, This means $x_2 = 0$ and $x_1$ is a free variable.

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The only eigenspace is one-dimensional, so there are not enough lin. ind eigenvectors for $A$ to be diagonalizable.