Worksheet 14

Sections 306 and 310
MATH 54

October 9, 2018

Exercise 1. For each of the following matrices, describe in geometric terms the real eigenspaces (if any) and their associated eigenvalues. Do not compute the matrices.

(a) The matrix induced by the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which reflects each vector across the z-axis.

(b) The matrix induced by the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates each vector by $\pi/4$ radians counterclockwise.

For part (b), discuss with your group how complex eigenvectors and eigenvalues may be involved.

For each of these parts, think about which vectors are scaled by T, or in other words "don't change angle" with the origin.

(a). - The z-axis is an eigenspace with eigenvalue 1, since vectors on the z-axis are not affected by the transformation.

- The x-y plane is an eigenspace with eigenvalue -1, since $(x,y,0)$ is sent to $(-x,-y,0)$.

(b). There are no real eigenspaces, since no vector is scaled by the transformation.

Exercise 2. Let T be defined by $T(\mathbf{x}) = A\mathbf{x}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$$

on pg 293,

By Thm. 8, if we write $A = PDP^{-1}$, and let β be the columns of P, then $[T]_\beta$ is diagonal. We first find the eigenvalues of A:

$\begin{vmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1) = 0$. So $\lambda = 3, 1$. We now find the eigenspaces:

$\underline{\lambda=1}$. We find the nullspace of $A - I1$: $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -3 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\lambda=3}$ Similarly, we solve $\left[\begin{array}{cc|c} -3 & 1 & 0 \\ -3 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

So $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1/3 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. So $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \right\}$.

Exercise 3. (a) Find eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$ of the following matrix:

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$

(b) Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form PCP^{-1}.

Part b on next page.

(a). To find the eigenvalues, we solve the characteristic equation:

$$\begin{vmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda^2 - 8\lambda + 15) + 2 = \lambda^2 - 8\lambda + 17 = 0 \qquad \text{So } \lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm i$$

$\underline{\lambda = 4+i}$: To find the eigenspace, we find the nullspace of $\begin{bmatrix} 5-(4+i) & -2 \\ 1 & 3-(4+i) \end{bmatrix}$

$\left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right]$. We know this has a nontriv solution, so since this a 2×2 matrix the second row must be some (complex) multiple of the first row. So we can solve the system using just the first equation. $(1-i)x_1 - 2(x_2) = 0$.

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{2}{1-i} \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$. So $\left\{ \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \right\}$ is a possible basis.

$\underline{\lambda = 4-i}$ This time, we find the null space of $\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}$. Similarly to above, we can just use the first row: $x_1(1+i) - 2x_2 = 0$.

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{2}{1+i} \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$. So $\left\{ \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \right\}$ is a possible basis.

Exercise 4. The following matrix is the matrix for a composition of a rotation and a scaling. Give the angle ϕ of rotation and the scalar factor r.

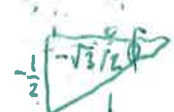
$$A = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$$

We pull out a factor of the magnitude of the first column:

$$r = \sqrt{(-\tfrac{\sqrt{3}}{2})^2 + (-\tfrac{1}{2})^2} = \sqrt{\tfrac{3}{4} + \tfrac{1}{4}} = 1.$$

$$\text{So } A = 1 \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix} = 1 \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

where ϕ is the angle given in the following tringle



From trigonometry, we see that $\phi = -210° = \frac{7\pi}{6}$ radians.

(b) By thm 9, on page 301. since $\otimes$. we have an eigenvalue of $4-i$ with eigenvector $\vec{v} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$,

We can write $A = PCP^{-1}$, where $P = \begin{bmatrix} \mathrm{Re}\,\vec{v} & \mathrm{im}\,\vec{v} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$,

and $C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$.

Exercise 5. True or false? Justify please! Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$.

(a) $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$

(b) $\text{dist}(\mathbf{u}, \mathbf{v}) + \text{dist}(\mathbf{v}, \mathbf{w}) = \text{dist}(\mathbf{u}, \mathbf{w})$

(a) True! This is because taking the dot product is commutative.

(b) False: Let $\vec{u} = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\0\end{bmatrix}$, $\vec{w} = \begin{bmatrix}0\\1\end{bmatrix}$.

counterexample

$\text{dist}(\vec{u}, \vec{v}) + \text{dist}(\vec{v}, \vec{w}) =$

$1 + 1 = 2$

but

$\text{dist}(\vec{u}, \vec{w}) = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Exercise 6. Show that if A is a real $n \times n$ matrix with $A^T = A$ and $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$ in $\mathbb{C}^n$, then λ is real. (This is a challenge! See 23 and 24 in section 5.5 for hints!)

It is hard for me to write out the notation by hand, but let me know if you have questions.

For hints, see #23, #24 in sec 5.5.