Worksheet 13

Sections 306 and 310
MATH 54

October 3, 2018

Exercise 1. Mark each statement True or False. Justify each answer. For these problems, A, B are $n \times n$ matrices.

(a) If A, B are row equivalent, then they have the same eigenvalues.

(b) If A has n eigenvectors, A is diagonalizable.

(c) If A has n distinct eigenvalues, it is diagonalizable.

(a). False! row operations can change eigenvalues! For example $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(b). False. A has to have n <u>linearly independent</u> eigenvectors

(c). True! There is a theorem saying that if λ_1, λ_2 are distinct eigenvalues with v_1, v_2 as corresponding eigenvectors, then $\{v_1, v_2\}$ is a linearly independent set.

32
128

Exercise 2. Find the characteristic polynomials and eigenvalues of the following matrices:

$$\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$$

The characteristic polynomial is

$$\begin{vmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = (\lambda^2 - 10\lambda + 21) + 4 =$$

$$\lambda^2 - 10\lambda + 25 = 0.$$

This factors to

$$(\lambda - 5)^2 = 0.$$

So $\lambda = 5$ is the only eigenvalue.

The characteristic polynomial is

$$\begin{vmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{vmatrix} = (\lambda^2 - 9\lambda + 20) + 12$$

$$= \lambda^2 - 9\lambda + 32 = 0$$

To find the eigenvalues, we use the quadratic formula:

$$\lambda = \frac{9 \pm \sqrt{81 - 128}}{2} = \frac{1}{2} \pm \frac{i\sqrt{47}}{2}.$$

Exercise 3. (a) As a group, discuss why it is useful to be able to diagonalize a matrix!

(b) If possible, diagonalize the following matrix:

$$\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$$

(b) We first find eigenvalues and eigenvectors.

To find eigenvalues, we solve the characteristic equation

$\begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 15 + 1 = (\lambda-4)^2$. So $\lambda = 4$ is the only eigenvalue. We now find the eigenspace by finding solutions to $\begin{bmatrix} 3-4 & -1 \\ 1 & 5-\lambda \end{bmatrix}\vec{x} = 0 \Rightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

So the eigenspace is $\text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$. Since there is only one eigenspace, and it is one dimensional, this matrix is not diagonalizeable.

Exercise 4. The eigenvalues of A are 2 and 8. Use this information to diagonalize A:

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

In order to diagonalize A, we need to find eigenspaces corresponding to each eigenvector.

$\underline{\lambda = 2}$: The eigenspace is solutions to $\begin{bmatrix} 4-2 & 2 & 2 \\ 2 & 4-2 & 2 \\ 2 & 2 & 4-2 \end{bmatrix}\vec{x} = \vec{0}$

$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ So the solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\underline{\lambda = 8}$: The eigenspace is solutions to $\begin{bmatrix} 4-8 & 2 & 2 \\ 2 & 4-8 & 2 \\ 2 & 2 & 4-8 \end{bmatrix}\vec{x} = \vec{0}$.

$\left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So the solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

So we have the following linearly independent set of 3 ~~eigenvectors~~ eigenvectors

$\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ with corresponding eigenvalues $\{2, 2, 8\}$

So using thm 5 on pg 284, $A = PDP^{-1}$ where $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

Exercise 5. Do A and A^T have the same characteristic polynomial?

Yes!

Note that the characteristic poly of A^T is

$$\det(A^T - \lambda I)$$

Using properties of transposes, we get:

$$\det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \det((A - \lambda I)^T) = \det(A - \lambda I)$$