Worksheet 12

Sections 306 and 310
MATH 54

October 2, 2018

Exercise 1. Mark each statement True or False. Justify each answer.

(a) The columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are linearly independent.

(b) If $V = \mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, then row reduction of $[\mathbf{c}_1 c_2 b_1 b_2]$ to $[IP]$ yields a matrix P that satisfies $[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{C}$ for all $\mathbf{x}$ in V.

(a) True! The columns of $P_{C \leftarrow B}$ are coordinate vectors of the $b_1, \dots b_n$. Coordinate vectors of lin. ind vectors are lin. ind.

(b). False. In the book, there's a discussion that

$P = P_{B \leftarrow C}$ is obtained by row reducing

$[\vec{b}_1 \; \vec{b}_2 \; \vec{c}_1 \; \vec{c}_2]$.

Exercise 2. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the bases $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis. Then write t as a linear combination of the polynomials in $\mathcal{B}$.
↖ sorry for the typo.

In this problem, we will denote the standard basis as $\mathcal{E} = \{1, t, t^2\}$.

Recall that the formula for $P_{\mathcal{E} \leftarrow \mathcal{B}} = \left[[\vec{b}_1]_{\mathcal{E}} \; [\vec{b}_2]_{\mathcal{E}} \; [\vec{b}_3]_{\mathcal{E}} \right]$

Since $\vec{b}_1 = 1 + 0t + -3t^2$, $[\vec{b}_1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$. Similarly,

$[\vec{b}_2]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$, $[\vec{b}_3]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Putting this all together,

we get: $P_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$.

We know that for any $\vec{p}$ in $\mathbb{P}_2$, $[p]_{\mathcal{E}} = P_{\mathcal{E} \leftarrow \mathcal{B}} [p]_{\mathcal{B}}$.

We want to find $[t]_{\mathcal{B}}$, and we know $[t]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Since change-of-coordinate matrices are invertible, we can rearrange the equation as

$[p]_{\mathcal{B}} = P_{\mathcal{E} \leftarrow \mathcal{B}}^{-1} [p]_{\mathcal{E}} = \begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ [illegible] & & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$

Exercise 3. (a) As a group, discuss the definitions of eigenvector and eigenvalue. Draw pictures!

(b) Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue.

(c) Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(b). To check if $\vec{x}$ is an eigenvector of A, we check to see if $A\vec{x} = \lambda\vec{x}$ for some scalar λ. In this case:

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

So $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector of the given matrix, with corresponding eigenvalue $\lambda = -2$.

(c). 3 is an eigenvalue of A iff and only if the equation $A\vec{x} = 3\cdot\vec{x}$ has a nontrivial solution. This rearranges to $(A - 3I)\vec{x} = 0$. We check if this has nontriv solutions using augmented matrices!

$$\left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So the system has only the triv solution. So $\lambda = 3$ is <u>not</u> an eigenvalue

Exercise 4. Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} \qquad \lambda = 4$$

To find the eigenspace, we find the solution set of $(A - \lambda I)\vec{x} = 0$

$$\left[\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 4 & -6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So the solutions are $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$.

So $\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}$ forms an eigenbasis for the $\lambda = 4$ eigenspace.

Exercise 5. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

The eigenvalues of an $n \times n$ matrix is exactly the set of roots of the characteristic polynomial, $\det(A - \lambda I) = 0$. By thinking about how cofactor expansion works, you can see that this will always be a degree n polynomial, which can have at most n distinct roots.

Exercise 6. Find the characteristic polynomial and the eigenvalues for the matrix.

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$$

The characteristic polynomial of a matrix A is $\det(A - \lambda I) = 0$. So we need to find

$$\det\left(\begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix}.$$

Since the middle row is almost all zeros, its easiest to do cofactor expansion on this row:

$$\begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix} = -0\begin{vmatrix} -2 & 3 \\ 7 & -2-\lambda \end{vmatrix} + (1-\lambda)\begin{vmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix} - 0\begin{vmatrix} 5-\lambda & -2 \\ 6 & 7 \end{vmatrix}$$

$$= (1-\lambda)\left[(5-\lambda)(-2-\lambda) - 18\right] = (1-\lambda)(\lambda^2 - 3\lambda - 10 - 18) =$$

$$(1-\lambda)(\lambda^2 - 3\lambda - 28) = (1-\lambda)(\lambda - 7)(\lambda + 4)$$

So the characteristic polynomial is $(1-\lambda)(\lambda-7)(\lambda+4) = 0$

and the eigenvalues are $\lambda = 1$, $\lambda = 7$, $\lambda = -4$.