Worksheet 11

Sections 306 and 310
MATH 54

September 27, 2018

Exercise 1. If a 6×3 matrix has rank 3, find dim Nul A, dim Row A, and rank A^T.

Note that rank A = dim col A = dim Row A.

So dim $\boxed{\text{Row } A = \text{rank } A = 3}$ Also, since the columns of A^T are the rows of A,

$\boxed{\text{rank } A^T = \dim \text{col } A^T = \dim \text{row } A = 3}$

To find dim Nul A, we use the rank theorem:

$$\text{rank } A + \dim \text{Nul } A = 3 \leftarrow \text{\# columns of } A.$$

So $\boxed{\dim \text{Nul } A = 0}$

Exercise 2. If A is a 6×4 matrix, what is the smallest possible dimension of Nul A?

By the rank theorem, rank A + dim Nul A = 4. Note that dim nul A is the smallest when rank A is the biggest. Since the columns of A are vectors in $\mathbb{R}^6$, it is possible for all four of them to be a lin. ind set. Since the largest possible value of rank A is 4, the smallest possible dimension of Nul A is 0.

Exercise 3. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be bases for a vector space V, and let $P = [[\mathbf{d}_1]_{\mathcal{A}}, [\mathbf{d}_2]_{\mathcal{A}}, [\mathbf{d}_3]_{\mathcal{A}}]$. Which of the following equations is true for all $\mathbf{x}$ in V?

(a) $[\mathbf{x}]_{\mathcal{A}} = P[\mathbf{x}]_{\mathcal{D}}$ See thm 15 on page 242.

(b) $[\mathbf{x}]_{\mathcal{D}} = P[\mathbf{x}]_{\mathcal{A}}$

Exercise 4. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for $\mathbb{R}^2$. Compute the change of coordinate matrix from $\mathcal{C}$ to $\mathcal{B}$.

$$b_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, b_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, c_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

There are several ways to do this. We are going to use a system of equations. Recall from the previous problem that $P_{\mathcal{B}\leftarrow\mathcal{C}} = [[\vec{c}_1]_{\mathcal{B}}, [\vec{c}_2]_{\mathcal{B}}]$.

Suppose $[\vec{c}_1]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\vec{c}_2]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

So $\vec{c}_1 = x_1\vec{b}_1 + x_2\vec{b}_2$ and $\vec{c}_2 = y_1\vec{b}_1 + y_2\vec{b}_2$.

To solve both systems at once, we row reduce the big augmented matrix.

$$[\vec{b}_1 \ \vec{b}_2 \mid \vec{c}_1 \ \vec{c}_2] = \left[\begin{array}{cc|cc} 7 & -3 & 1 & -2 \\ 5 & -1 & -5 & 2 \end{array}\right] \Longrightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 3 \end{bmatrix}$$

I skipped some steps, since we are all row reduction experts now.

So $P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} -2 & 1 \\ -3 & 3 \end{bmatrix}$.