# Worksheet 10

Sections 306 and 310
MATH 54

September 25, 2018

**Exercise 1.** Assume that $A$ is row equivalent to $B$. Find bases for nul $A$ and col $A$.

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & 3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

nul A: By definition, nul A is the set of solutions to $A\vec{x} = \vec{0}$. We look at the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & -5 & 11 & 3 & 0 \\ 2 & 4 & -5 & 15 & 2 & 0 \\ 1 & 2 & 0 & 4 & 5 & 0 \\ 3 & 6 & -5 & 19 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 5 & 0 \\ 0 & 0 & 5 & -7 & 8 & 0 \\ 0 & 0 & 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Putting the solution in parametric vector form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}$$

These two vectors are lin ind, so they form a basis of the null space

col A: By definition, col A is the span of the columns of A. A basis of col A is given by the pivot columns of A. From B, we see that the 1st, 3rd, and 5th columns are pivot columns. So a basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$$

**Exercise 2.** True or false? Give brief justifications.

(a) A linearly independent set in a subspace $H$ is a basis for $H$.

(b) If a finite set $S$ of nonzero vectors spans a vector space $V$, then some subsets of $S$ is a basis of $V$.

(c) If $B$ is an echelon form of a matrix $A$, the pivot columns of $B$ for a basis of col $A$.

(a) False. To be a basis, the set also has to span H. For example, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a linearly independent set, but it is not a basis of $\mathbb{R}^3$ since it doesn't span $\mathbb{R}^3$.

(b) True. See the spanning set theorem.

(c) False. B tells you which of the columns are pivot columns, but to find a basis of the column space you need to choose the pivot columns of the original matrix A.

**Exercise 3.** Find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_\beta$ and the given basis $\beta$.

$$\beta = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\} \qquad [\mathbf{x}]_\beta = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

Since $[\vec{x}]_\beta = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$, $\vec{x} = 8\begin{bmatrix} 4 \\ 5 \end{bmatrix} - 5\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

**Exercise 4.** Find the coordinate vector $[\mathbf{x}]_\beta$ of $\mathbf{x}$ relative to the given basis $\beta$.

$$\beta = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\} \qquad \mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We want to find $[\vec{x}]_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ In other words, we want to find $x_1, x_2$ such that

$x_1\begin{bmatrix} 1 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. So we want to solve the system

$$\begin{aligned} x_1 + 2x_2 &= -2 \\ -3x_1 - 5x_2 &= 1 \end{aligned}$$

We set up the following augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ -3 & -5 & 1 \end{array}\right] \xrightarrow{3R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -5 \end{array}\right] \xrightarrow{-2R_2 + R_1 \to R_1}$$

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -5 \end{bmatrix}$$

So $x_1 = 8$
$x_2 = -5$.

So $[\vec{x}]_\beta \overset{2}{=} \begin{bmatrix} 8 \\ -5 \end{bmatrix}$.

**Exercise 5.** Find a basis of the following vector spaces. What is the dimension of each?

$$\left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\} \qquad \{(a,b,c,d) : a - 3b + c = 0\}$$

$H = \left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s,t \text{ in } \mathbb{R} \right\} = \left\{ s\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} : s,t \text{ in } \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

by the definition of span. $\left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a set of 2 lin. ind. vectors that span H, so they form a basis of H. So H has dimension 2.

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$H = \{(a,b,c,d) : a-3b+c=0\}$ is the solution set of the equation $a-3b+c=0$

Putting this in parametric vector form, we get $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = b\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

So $H = \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (Exercise to reader; check that these 3 are lin ind).

Since $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a lin ind. set of vectors that span H, it is a basis. So H has dim 3

**Exercise 6.** Let $T : V \to W$ be a linear transformation. Shot that if $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is linearly dependent $V$, then $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_p})\}$ is linearly dependent in $W$. Use this to show that if $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_p})\}$ is linearly independent in $W$, then $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is linearly independent in $V$.

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Suppose $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is lin dependent. Then there exist $c_1, \ldots, c_p$ such that $c_1\vec{v}_1 + \ldots + c_p\vec{v}_p = \vec{0}$, and not all of $c_1, \ldots, c_p$ are 0.

Apply T to both sides of the equation to get

$T(c_1\vec{v}_1 + \ldots + c_p\vec{v}_p) = T(\vec{0})$.

Using properties of lin. transformations, this simplifies to

$c_1 T(\vec{v}_1) + \ldots + c_p T(\vec{v}_p) = 0$.

Recall from above that not all of $c_1, \ldots, c_p$ are 0.

So by the definition of lin dep, $\{T(\vec{v}_1), \ldots T(\vec{v}_p)\}$ is lin dep.

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Suppose that $\{T(\vec{v}_1) \ldots, T(\vec{v}_p)\}$ is lin, ind. in W. Assume for the sake of contradiction that $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is lin. dep. But then, by the previous discussion, $\{T(\vec{v}_1) \ldots, T(\vec{v}_p)\}$ is lin dep, which contradicts our statement above. So by contradiction, $\{\vec{v}_1, \ldots, \vec{v}_p\}$ must be linearly independent.