Worksheet 9

Sections 207 and 219
MATH 54

Febrary 21, 2018

Exercise 1. Determine if the following sets of polynomials are subspaces of the space $\mathbb{P}_3$ of polynomials in t of degree at most 3.

a. All polymonials in $\mathbb{P}_3$ of the form $p(t) = at^2$, where a is in $\mathbb{R}$. Yes!

b. All polynomials in $\mathbb{P}_3$ of the form $a + t^2$, where a is in $\mathbb{R}$. No!

c. All polynomials p in $\mathbb{P}_3$ such that $p(0) = 0$. Yes!

(a). This can be written as span$\{t^2\}$, and the span of a finite set is always a subspace of the bigger space.

(b). No! This set is not closed under additon. Consider, for example, $5+t^2$ and $6+t^2$. These are both in the set, but their sum, $11+2t^2$ is not, since it is not of the rright form.

(c). There are many ways to do this, here I will show this is a subspace by showing it satisfies the definition of a subspace.

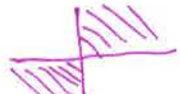
(i). First, the 0-polynomial, $p(t)=0$, indeed sends 0 to 0. So the 0-polynomial is in this set.

(ii) We now show this set is closed under addition. Suppose $p(t), q(t)$ are in this set. (Thus $p(0)=0$, $q(0)=0$). We now show that there sum, ~~p(t)+q(t) is~~ $p+q(t)$ is in the set:
$p+q(0) = p(0)+q(0) = 0+0=0$. so $p+q$ is indeed in the set.

(iii). We now show this set is closed under scalar multiplication. Suppose c is a scalar, and $p(t)$ is in the set. (so $p(0)=0$) We check to see whether $cp(t)$ is in the set.
~~cp(0)=0~~ $cp(0) = c(0) = 0$ so $cp(t)$ is in the set.

The set satisfies the definition of a subspace :)

Exercise 2. Let W be the union of the first and third quadrants of the plane. That is: let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$.



(a) If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

(b) Can you find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that their sum is not in W.

(c) Is W a vector space?

(a) Yes. Suppose $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ is in W, so $xy \geq 0$. We will see that $c\vec{u}$ is also in W. $c\vec{u} = c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$. We check the product of the entries:

$$(cx)(cy) = c^2 xy \geq 0, \text{ since } xy \geq 0 \text{ and } c^2 \geq 0.$$

So $c\vec{u}$ is indeed in W.

(b) Yes! $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are in W, but their sum $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is not.

(c) No it is not! It sits inside $\mathbb{R}^2$ but does not satisfy the definition of being a subspace of $\mathbb{R}^2$.

Exercise 3. For each of the following sets, either use an appropriate theorem to show that the given set is a vector space, or find an specific example to the contrary.

$$\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 5r - 1 = s + 2t \right\}$$

$$\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a + 3b = c \\ b + c + a = d \end{matrix} \right\}$$

No! This does not contain the 0-vector, since setting $r=0, s=0, t=0$ does not satisfy the condition

$$5r - 1 = s + 2t.$$

Since a vector space must contain a zero vector, this is not a vector space.

We can rearrange the equations and see that this set is the set of solutions of

$$a + 3b - c = 0$$
$$b + c + a - d = 0$$

So this is the null space of

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Null spaces are always vector spaces.

Exercise 4. Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. Is $\mathbf{w}$ in Col A? Is it in Nul A?

"Is $\vec{w}$ in col A?" is the same question as "is $A\vec{x} = \vec{w}$ consistent?".

We look at the augmented matrix

$$\left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -1 \end{array}\right] \xrightarrow{\text{row operations}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This is inconsistent, so $\vec{w}$ is not in col A.

"Is $\vec{w}$ in nul A?" is the same question as "Is $A\vec{w} = 0$?".

We check this:

$$A\vec{w} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -16 - 2 + 9 \\ 12 + 4 - 8 \\ 8 - 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \\ 4 \end{bmatrix}$$

Since $A\vec{w} \neq 0$, $\vec{w}$ is not in nul A.

Exercise 5. True or false? Justify each answer:

(a) A null space is a vector space. Yes! The nullspace of a $n \times m$ matrix is a subspace of $\mathbb{R}^m$. Subspaces are vector spaces.

(b) The column space of an $m \times n$ matrix is in $\mathbb{R}^m$. True

(c) Col A is the set of all solutions of $A\mathbf{x} = \mathbf{b}$. False.

(d) Nul A is the kernel of the mapping $\mathbf{x} \mapsto A\mathbf{x}$. True.

(e) The range of a linear transformation is a vector space. True.

(b)

An $m \times n$ matrix looks like:

$$m\left\{\underbrace{[\;]\,[\;]\cdots[\;]}_{n}\right.$$

The columns do have length m, so col A is a subspace of $\mathbb{R}^m$.

(c)

The solution set of $A\vec{x} = \vec{b}$ isn't always even a subspace, and col A always is. Thus, these cannot be the same.

(d)

The definition of kernel of $\vec{x} \mapsto A\vec{x}$ is the set of all $\vec{x}$ such that $A\vec{x} = 0$. This is the same as the definition of nul A.

(e)

The range of the lin transf. $T(\vec{x}) = A\vec{x}$ is the same as col A. The column space of a matrix is always a vector space.