Worksheet 8

Sections 207 and 219
MATH 54

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Exercise 1. Determine the values of s such that the system has a unique solution. Use Cramer's rule to describe the solutions in terms of s.

$$3sx_1 + 5x_2 = 3$$
$$12x_1 + 5sx_2 = 2$$

We rewrite this as $\begin{bmatrix} 3s & 5 \\ 12 & 5s \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

A matrix equation $A\vec{x} = \vec{b}$ has a unique solution if and only if A is invertible, i.e. $\det A \neq 0$. Here, $\det A = 15s^2 - 60 = 15(s^2-4) = 15(s-2)(s+2)$.
So $\det A \neq 0$ precisely when $s \neq 2, -2$.

Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the unique solution of the system (supposing one exists).
By cramer's rule,

$$x_1 = \frac{\begin{vmatrix} 3 & 5 \\ 2 & 5s \end{vmatrix}}{\begin{vmatrix} 3s & 5 \\ 12 & 5s \end{vmatrix}} = \frac{15s-10}{15s^2-60} = \frac{3s-2}{3s^2-12}.$$

$$x_2 = \frac{\begin{vmatrix} 3s & 3 \\ 12 & 2 \end{vmatrix}}{\begin{vmatrix} 3s & 5 \\ 12 & 5s \end{vmatrix}} = \frac{6s-36}{15s^2-60} = \frac{2s-12}{5s^2-20}.$$

Exercise 2. Compute the adjugate of the given matrix, and use theorm 8 to give the inverse of the matrix:

$$\begin{bmatrix} 1 & 1 & 3 \\ -2 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

We use the following formula to find the adjugate:

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} \end{bmatrix}^T =$$

$$\begin{bmatrix} -3 & 2 & -2 \\ 2 & 1 & -1 \\ 7 & -7 & 0 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 & 7 \\ 2 & 1 & -7 \\ -2 & -1 & 0 \end{bmatrix}$$

$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$. So first we find $\det A$ using cofactor expansion along the first column.
$\det A = 1\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$ which using work we already did is $1(-3) - 2(2) = -7$.

So $A^{-1} = -\frac{1}{7} \begin{bmatrix} -3 & 2 & 7 \\ 2 & 1 & -7 \\ -2 & -1 & 0 \end{bmatrix}$

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Exercise 3. Suppose that all the entries in A are integers. Are the entries in A^{-1} necesarily integers? What if $\det(A) = 1$? Explain.

First question:

No! See Exercise 2 for a counterexample :-)

The issue is that when you divide by the determinant in the formula $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$, you can get entries that are not integers.

Second question

If $\det(A) = 1$, Then entries are all integers. The formula for A^{-1} becomes: $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{1} \operatorname{adj} A = \operatorname{adj} A$.

To get the entries in adj A, you only use $+, -, \times$ operations on the original entries, so you cannot get anything that is not an integer.

Exercise 4. Find the volume of the parallelepiped with one vertix at the origen and adjacent vertices at (1,3,0), (-2,0,2), and (-1,3,-1).

The formula for the volume of such a parallelipiped is

$$\left| \begin{vmatrix} 1 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{vmatrix} \right| = \left| 1 \begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} - 3 \begin{vmatrix} -2 & -1 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} -2 & -1 \\ 0 & 3 \end{vmatrix} \right| =$$

You put the defining vertices as the columns of the matrix

$$\left| (0-6) - 3(2+2) \right| = \left| -6-12 \right| = 18.$$

Exercise 5. Let S be the parallelogram determined one vertex at the origin and adjacent vertices at (-2,3) and (-2,5). Let $A = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$. Try computing in two different ways!.

Method 1:

Our parallelogram has vertices (0,0), (-2,3), (-2,5), (-4,8).

our defining vertices are adjacent to the origin.

The new defining vertices (after the transformation) are $\begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -21 \\ 12 \end{bmatrix}$ and $\begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -27 \\ 16 \end{bmatrix}$

Thus the area of this new parallelogram is

$\left|\begin{vmatrix} -21 & -27 \\ 12 & 16 \end{vmatrix}\right| = |-21(16) + 12(27)|$

$= |-12(28-27)| = |-12| = 12.$

Method 2.

The area of our original parallelogram is $\left|\begin{vmatrix} -2 & -2 \\ 3 & 5 \end{vmatrix}\right| = |-10+6| = 4.$

According to a theorem in this section, Area (T(S)) = |det A × area of S|, where A is the matrix that defines the linear transformation T.

Using this, our area is:

$\left|\begin{vmatrix} 6 & -3 \\ -3 & 2 \end{vmatrix} \cdot 4\right| = |3 \cdot 4| = 12.$

Yay we get the same thing both ways!

Exercise 6. Find a basis of Nul(A) by listing vectors that span the null space:

$$\begin{bmatrix} 1 & 6 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

How many entries do vectors in the null space have? How many entries to vectors in the column space have? What is the dimension of each space?

We first find Nul (A) by finding the solution set of:

$\left[\begin{array}{ccccc|c} 1 & 6 & -4 & -3 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-6R_2+R_1 \to R_1}$

$\left[\begin{array}{ccccc|c} 1 & 0 & 8 & -9 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

The solution set in parametric vector form is:

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -8 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 9 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

So $\begin{bmatrix} -8 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis

vectors in null space have 5 entries

vectors in col space have 3 entries

dim col A = 2

dim nul A = 3.