Worksheet 7

Sections 207 and 219
MATH 54

February 12, 2019

Exercise 1. Compute the determinant in your favorite way.

$$\begin{vmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -3 & -5 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & -1 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & -3 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{vmatrix}$$

$$= 3\,(-8) = -24$$

the det of a triangular matrix is the product of the diagonals.

Exercise 2. Suppose that we already know that:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

Compute the following determinant:

$$\begin{vmatrix} -5d+g & -5e+h & -5f+i \\ a & b & c \\ g & h & i \end{vmatrix}$$

$$\begin{vmatrix} -5d+g & -5e+h & -5f+i \\ a & b & c \\ g & h & i \end{vmatrix} = \begin{vmatrix} -5d & -5e & -5f \\ a & b & c \\ g & h & i \end{vmatrix} =$$

$$-5\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = 5\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5 \cdot 7 = 35.$$

Exercise 3. Determine the values of s such that the system has a unique solution. Use Cramer's rule to describe the solutions in terms of s.

$$3sx_1 + 5x_2 = 3$$

$$12x_1 + 5sx_2 = 2$$

$12x_1 + 5sx_2 = 2$

We rewrite this as $\begin{bmatrix} 3s & 5 \\ 12 & 5s \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. This has exactly one solution if and only if $\begin{bmatrix} 3s & 5 \\ 12 & 5s \end{bmatrix}$ is invertible, which is the case if and only if $\begin{vmatrix} 3s & 5 \\ 12 & 5s \end{vmatrix} = 15s^2 - 60 = 15(s+2)(s-2) \neq 0$.

So this has a unique solution if and only if $s \neq \pm 2$.

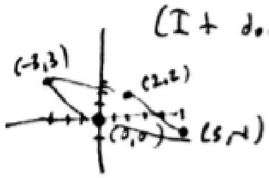
By cramers rule, $x_1 = \dfrac{\begin{vmatrix} 3 & 5 \\ 2 & 5s \end{vmatrix}}{\det A} = \dfrac{15s-10}{15s^2-60}$

$$x_2 = \frac{\begin{vmatrix} 3s & 3 \\ 12 & 2 \end{vmatrix}}{\det A} = \frac{6s-36}{15s^2-60}.$$

Exercise 4. Find the area of a parallelogram whose vertices are listed: (0,-2), (5,-3), (-3,1), (2,0).

We first draw the parallelogram  (this is a parellogram I promise :-))

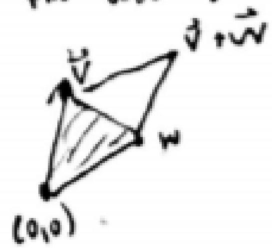
In order to use the determinant formula for area, we have to move it so that one of the vertices is at the origen. One way to do this is shifting it up by 2. (It doesn't matter which vertex you choose to move to the origin.



The two vertices adjacent to the origin are $(5,-1)$ and $(-3,3)$. So by the determined formula for area we have: Area $= \left|\det\begin{bmatrix}5 & -3\\ -1 & 3\end{bmatrix}\right| = 12.$

Exercise 5. Find the area of a triangle whose vertices are $(0,0)$, (v_1, v_2), (w_1, w_2)

From our parallelogram formula, we know that the area of the following parallelogram is $\left|\det\begin{bmatrix}v_1 & w_1\\ v_2 & w_2\end{bmatrix}\right|$



The triangle we are interested in is exactly half this area.

So the area is $\frac{1}{2}\left|\det\begin{bmatrix}v_1 & w_1\\ v_2 & w_2\end{bmatrix}\right|$

Exercise 6. If A is a 3×4 matrix, what is the smallest possible dimension of nul A?

Proof. Note that dim nul A + col $A = 4$. Since the dimension of the nul space and the dimension of the column space sum to some constant number, the dimension of the nul space is smallest precisely when co A is largest. Col A can be 3 at maximum, since the columns of A are of length 3 and cannot space a space of dimension larger than dim $\mathbb{R}^3$. So the dimension of nul A has to be at least 1. □

Exercise 7. Let $A = \begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix}$. Write $5A$. Is $\det(5A) = 5\det(A)$? Let A be a $n \times n$ matrix and let k be a scalar. Find a formula for $\det(kA)$ in terms of k and $\det(A)$.

$\det A = 6 - 4 = 2.$

$$\det 5A = \begin{vmatrix} 15 & 5 \\ 20 & 10 \end{vmatrix} = 150 - 100 = 50.$$

So $5 \det A \neq \det 5A$.

It turns out that $\det(kA) = k^n \det(A)$.

We can use the fact that $\det(AB) = \det(A)\det(B)$ to show this.

Note that $kA = \begin{bmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{bmatrix} A.$

k's on diagonal, 0's everywhere else

So $\det(kA) = \det \begin{bmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{bmatrix} \det A = k^n \det(A).$

the determinant of a diagonal matrix is the product of the diagonals.