Worksheet 6

Sections 207 and 219
MATH 54

February 7, 2019

Exercise 1. Let A be an $n \times n$ matrix. Suppose the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution for some $\mathbf{b}$ in $\mathbb{R}^n$. Can the columns of A span $\mathbb{R}^n$?

Proof. No. We use the invertible matrix theorem, on page 114 (thm. 8) If an $n \times n$ matrix has more than one solution to $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{b}$, then A cannot be invertible. If a square matrix is not invertible, then its columns cannot span $\mathbb{R}^n$. $\square$

Exercise 2. Determine which of the matrices are invertible. Justify your answers, but try using as few calculations as possible :)

$$\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix} \qquad \begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$$

Not invertible!

The determinant is $(-4)(-9) - 6 \cdot 6 = 0$.

See Thm 4 on page 173.

$$\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$$

Not invertible! There is a column of all 0's, so the columns do not form a lin. ind. set.

See part (e) of Thm 8 on page 114.

$$\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Invertible! This is already in row-echelon form, and we can see there are 4 pivot positions.

See part (c) of Thm 8 on page 114.

Exercise 3. Is it possible for a 5×5 matrix to be invertible when its columns do not span $\mathbb{R}^5$? Why or why not?

It is not possible.

By part (h) of Thm. 8 on page 114, a 5×5 matrix is invertible if and only if its columns span $\mathbb{R}^5$.

Exercise 4. Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. Is $\mathbf{w}$ in Col A? Is it in Nul A?

Col A.

To check if $\vec{w}$ is in col A, we check if $A\vec{x} = \vec{w}$ has a solution:
We can do this by reducing the following augmented matrix.

$$\begin{bmatrix} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -1 \end{bmatrix} \xrightarrow{\text{(I skipped some steps)}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is inconsistent, so $A\vec{x} = \vec{w}$ doesn't have a solution.

So $\vec{w}$ is not in col A.

$A\vec{w} = \begin{bmatrix} -9 \\ 8 \\ 4 \end{bmatrix}$ This is not $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $\vec{w}$ is not in the null space.

bases for $\mathbb{R}^3$. Justify your answers

Exercise 5. Determine which of the following sets are bases for $\mathbb{R}^3$. Justify your answers.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \begin{bmatrix} -4 \\ -5 \\ 6 \end{bmatrix}$$

Discuss with your group: Do you think that a set of two vectors can form a basis for $\mathbb{R}^3$? Why or why not? (We will discuss the idea of dimension soon, get excited!!)

(a). Not a basis! Not linearly independent.

(b) A basis. We can row reduce the matrix $\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$ to the identity $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By the invertible matrix theorem, this has lin ind. columns and these columns span $\mathbb{R}^3$.

(c). Not a basis. The vectors do not span $\mathbb{R}^3$. To see this, we look at $\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \vec{x} = \vec{b}$.
$\begin{bmatrix} 1 & -4 & b_1 \\ 2 & -5 & b_2 \\ -3 & 6 & b_3 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & c_3 \end{bmatrix}$ which is not always consistent.

It turns out that all bases of $\mathbb{R}^3$ have 3 vectors, we will talk about this later!

Exercise 6. Assume that A is row equivalent to B. Find bases for nul A and col A.

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & 3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & 3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

nul A: By definition, nul A is the set of solutions to $A\vec{x} = \vec{0}$. We look at the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & -5 & 11 & 3 & 0 \\ 2 & 4 & -5 & 15 & 2 & 0 \\ 1 & 2 & 0 & 4 & 5 & 0 \\ 3 & 6 & -5 & 19 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 5 & 0 \\ 0 & 0 & 5 & -7 & 8 & 0 \\ 0 & 0 & 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Putting the solution in parametric vector form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}$$

These two vectors are lin ind, so they form a basis of the null space

col A: By definition, col A is the span of the columns of A. A basis of col A is given by the pivot columns of A. From B, we see that the 1st, 3rd, and 5th columns are pivot columns. So a basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$$

Exercise 2. True or false? Give brief justifications.

(a) A linearly independent set in a subspace H is a basis for H.

(b) If a finite set S of nonzero vectors spans a vector space V, then some subsets of S is a basis of V.

(c) If B is an echelon form of a matrix A, the pivot columns of B for a basis of col A.

Exercise 7. True or false? Give brief justifications.

(a) A linearly independent set in a subspace H is a basis for H.

(b) If a finite set S of nonzero vectors spans a vector space V, then some subsets of S is a basis of V.

(c) If B is an echelon form of a matrix A, the pivot columns of B form a basis of col A.

(a) False. To be a basis, the set also has to span H. For example, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a linearly independent set, but it is not a basis of $\mathbb{R}^3$ since it doesn't span $\mathbb{R}^3$.

(b) True. See the spanning set theorem.

(c) False. B tells you which of the columns are pivot columns, but to find a basis of the column space you need to choose the pivot columns of the original matrix A.