Worksheet 5

Sections 207 and 219
MATH 54

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Exercise 1. Show that if the columns of B are linearly dependent, then so are the columns of AB.

Suppose the columns of B are lin. dependent. Then there is some nontrivial $\vec{x}$ such that $B\vec{x} = 0$. Now let's try multiplying AB by this same $\vec{x}$:

$$(AB)\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}$$

So there is nontrivial $\vec{x}$ such that $(AB)\vec{x} = \vec{0}$.

Thus, the columns of AB are linearly dependent.

Exercise 2. Find matrices A, B, C, such that $AB = AC$, yet $B \neq C$.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ $C = \begin{bmatrix} 1 \\ 5{,}000 \end{bmatrix}$.

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 5{,}000 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So $AB = AC$ but $B \neq C$.

Exercise 3. Solve the system using matrix inverses!

$$8x_1 + 5x_2 = -9$$

$$-7x_1 - 5x_2 = 11$$

The matrix system is $A\vec{x} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$, where $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$

Multiplying both sides of the equation by A^{-1}, we get:

$\vec{x} = A^{-1}\begin{bmatrix} -9 \\ 11 \end{bmatrix}$. Thus, to find $\vec{x}$, we just have to comput A^{-1}. Using the formula for 2x2 matrices,

$A^{-1} = \frac{1}{-40+35}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = 1\,\frac{1}{5}\begin{bmatrix} 5 & 5 \\ -7 & -8 \end{bmatrix}$. Plugging this into our formula for $\vec{x}$, we get: $\vec{x} = \frac{1}{5}\begin{bmatrix} 5 & 5 \\ -7 & -8 \end{bmatrix}\begin{bmatrix} -9 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Exercise 4. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is an invertible $n \times n$ matrix. show that $B = C$. Is this necesarily true if D is not invertible?

If D is invertible, we can multiply both sides by D^{-1}.

$$(B-C)DD^{-1} = 0D^{-1}$$
$$B-C=0$$
$$B=C.$$

So B and C must be equal.

This is not nec. true if D is not invertible!!

We can use similar matrices as in ex 2.

$$\left(\begin{bmatrix}1\\5\end{bmatrix} - \begin{bmatrix}1\\5{,}000\end{bmatrix}\right)\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0\\-4995\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = 0, \text{ but}$$
$$\begin{bmatrix}1\\5\end{bmatrix} \neq \begin{bmatrix}1\\5{,}000\end{bmatrix}$$

Exercise 5. Explain why the columns of an $n \times n$ matrix A span $\mathbb{R}^n$ when A is invertible.

Suppose A is invertible. Then $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^n$.
(since you can multiply both sides by A^{-1} and get $\vec{x} = A^{-1}\vec{b}$).

We now reinterpret this matrix equation as a vector equation:

Let $\vec{a}_1, \ldots, \vec{a}_n$ be the columns of A. Then, we reinterpret our above statement to say:

$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^n$.

in other words, every vector in $\mathbb{R}^n$ is in the span of $\vec{a}_1, \ldots, \vec{a}_n$.

So $\vec{a}_1, \ldots, \vec{a}_n$ indeed span $\mathbb{R}^n$.

63-88 = -25

Exercise 6. In this exercise, we prove that the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists if and only if $ad - bc \neq 0$. In the case that it does exist, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(a) Show that if $ad - bc = 0$, then the equation for $A\mathbf{x} = \mathbf{0}$ has more than one solution. Conclude that then A must not be invertible.

(b) Show that if $ad - bc \neq 0$, then the formula holds.

(a). We do some row operations to put this in an echelon form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[aR_2]{cR_1} \begin{bmatrix} ac & bc \\ ca & da \end{bmatrix} \xrightarrow{-R_1+R_2 \to R_2} \begin{bmatrix} ac & bc \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} ac & bc \\ 0 & 0 \end{bmatrix}$$

Since we were given $ad-bc=0$.

Since there is no pivot in the second column, $A\vec{x}=\vec{0}$ has ∞-many solutions. Thus, A can not be inv. since if A was invertible, $A\vec{x}=\vec{0}$ would have exactly one solution.

(b). If $ad-bc\neq 0$, the formula is plausible, since $\frac{1}{ad-bc}$ is defined. We now verify that $A\left[\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$ indeed gives us the identity matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ba+ba \\ cd-cd & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

One should also check that $\left[\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]A = I$.

Thus, A^{-1} is indeed $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$