Worksheet 4

Sections 207 and 219
MATH 54

January 31, 2019

Exercise 1. For each pair $A, \mathbf{b}$, let T be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. For each, find a vector whose image under T is $\mathbf{b}$. Is this vector unique?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Another way to phrase this question is: "Find an $\vec{x}$ such that $T\vec{x} = \vec{b}$. Is this $\vec{x}$ the only solution to the matrix equation?".

The augmented matrix of $T\vec{x} = \vec{b}$ is:

$$\begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{bmatrix} \xrightarrow[-3R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{2R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix} \xrightarrow{\frac{1}{5}R_3} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[2R_3+R_1 \to R_1]{-2R_3+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{So } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

is mapped to $\vec{b}$ by T. Since there are no free variables, this is the only such vector.

The augmented matrix of $T\vec{x} = \vec{b}$ is:

$$\begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{bmatrix} \xrightarrow{3R_1+R_2 \to R_2} \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -8 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{8}R_2} \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{5R_2+R_1 \to R_1}$$

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$ So any $\vec{x}$ of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is mapped to $\vec{b}$. So $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is such a vector, and it is not unique.

Exercise 5. Describe geomet[illegible]ally what the following linear transformation T does. It

Exercise 2. Describe geometrically what the following linear transformation T does. It may be helpful to plot a few points and their images!

$$T = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .5x_1 \\ x_2 \end{bmatrix}$

So T contracts the first coordinate of a point by $\frac{1}{2}$, and preserves the second coordinate.

Exercise 3. Let $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y_1} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$ and $\mathbf{y_2} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e_1}$ to $\mathbf{y_1}$ and $\mathbf{e_2}$ to $\mathbf{y_2}$. What is the image of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$?

We know that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

So $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x_1\begin{bmatrix} 1 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

by the definition of a lin trans.

Exercise 4. Show that $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ is a linear transformation.

In order to show that T is a linear transformation we need to show that $T(a\vec{u}+b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$ for any $\vec{u}, \vec{v}$ in $\mathbb{R}^2$, a, b in $\mathbb{R}$.

Let $\vec{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then:

$$T(a\vec{u} + b\vec{u}) = T\left(a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}\right) =$$

$$\begin{bmatrix} ax_2 + by_2 \\ ax_1 + bx_1 \end{bmatrix} = a\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + b\begin{bmatrix} y_2 \\ y_1 \end{bmatrix} = aT\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + bT\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) =$$

$aT(\vec{u}) + bT(\vec{v})$, as desired.

So T is indeed a linear transformation.

Exercise 5. Assume T is a linear transformation. Find the standard matrix of T.

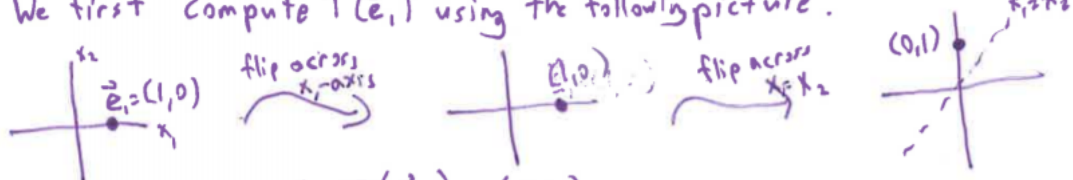
- $T : \mathbb{R}^3 \to \mathbb{R}^2$, and $T(\mathbf{e_1}) = (1, 3)$, $T(\mathbf{e_2}) = (4, -7)$, $T(\mathbf{e_3}) = (-4, 5)$, where $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$ are the columns of the 3×3 identity matrix.
- $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1- axis and then reflects points through the line $x_1 = x_2$.
- $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $T(x_1, x_2) = (x_1 - x_2, -2x_1 + x_2, x_1)$.

As a group, choose one of these transformations and figure out if it is one-to-one and onto.

- Since $T(\vec{e}_1) = (1,3)$, $T(\vec{e}_2) = (4,-7)$, $T(\vec{e}_3) = (-4,5)$, the standard matrix is:

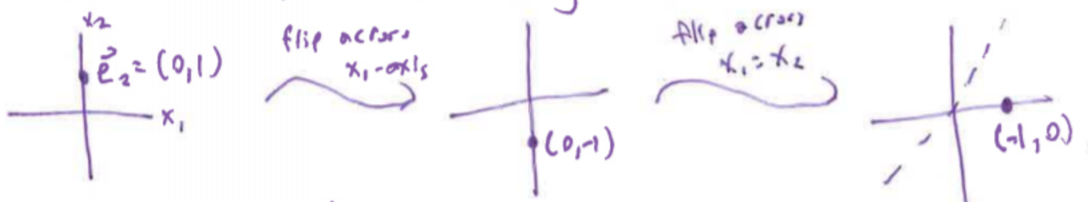
$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 4 & -4 \\ 3 & -7 & 5 \end{bmatrix}$$

- We first compute $T(\vec{e}_1)$ using the following picture:



So $T(\vec{e}_1) = (0,1)$.

We now compute $T(\vec{e}_2)$ using the following picture:



So $T(\vec{e}_2) = (-1,0)$.

So the standard matrix is $A = [T(\vec{e}_1), T(\vec{e}_2)] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- We compute $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

$T(\vec{e}_1) = T(1,0) = (1-0, -2+0, 1) = (1,-2,1)$.

$T(\vec{e}_2) = T(0,1) = (0-1, 0+1, 0) = (-1,1,0)$.

So the standard matrix is $A = [T(\vec{e}_1), T(\vec{e}_2)] = \begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}$

Worksheet 5