Worksheet 3

Sections 207 and 219
MATH 54

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Exercise 1. Write the following vector equation as a matrix equation and also as a system of linear equations. Solve the system and express your answer in parametric vector form.

$$x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Matrix Equation: $\begin{bmatrix} -2 & 8 & 1 \\ 3 & 5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

linear system: $-2x_1 + 8x_2 + x_3 = 0$
$3x_1 + 5x_2 - 6x_3 = 0$

To solve, we row-reduce the augmented matrix.

$$\begin{bmatrix} -2 & 8 & 1 & 0 \\ 3 & 5 & -6 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & -4 & -\frac{1}{2} & 0 \\ 3 & 5 & -6 & 0 \end{bmatrix} \xrightarrow{-3R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -4 & -\frac{1}{2} & 0 \\ 0 & 17 & -\frac{9}{2} & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{17}R_2 \to R_2} \begin{bmatrix} 1 & -4 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{9}{34} & 0 \end{bmatrix} \xrightarrow{-4R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & -\frac{53}{34} & 0 \\ 0 & 1 & -\frac{9}{34} & 0 \end{bmatrix}.$$

So the solutions satisfy $x - \frac{53}{34} z = 0$, $y - \frac{9}{34} z = 0$

Exercise 2. Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$.

$$\mathbf{a_1} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{a_2} = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \mathbf{a_3} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ 8 \end{bmatrix}$$

b is a linear combination of a_1, a_2, a_3 if and only if there exist x_1, x_2, x_3 such that:

$$x_1\begin{bmatrix}1\\-2\\2\end{bmatrix} + x_2\begin{bmatrix}0\\5\\5\end{bmatrix} + x_3\begin{bmatrix}2\\0\\8\end{bmatrix} = \begin{bmatrix}-5\\11\\8\end{bmatrix}.$$

The augmented matrix of this vector equation is

$$\begin{bmatrix}1 & 0 & 2 & -5\\-2 & 5 & 0 & 11\\2 & 5 & 8 & 8\end{bmatrix}$$ which reduce to $$\begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & \frac{4}{5} & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

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This is inconsistent.
So b is not a linear combination of a_1, a_2, a_3.

Exercise 3. Write the following products as linear combinations of the columns of the matrix. Use this to compute the product.

$$\begin{bmatrix}6 & 5\\-4 & -3\\7 & 6\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix} \qquad \begin{bmatrix}8 & 3 & -4\\5 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}6 & 5\\-4 & -3\\7 & 6\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix} =$$

$$2\begin{bmatrix}6\\-4\\7\end{bmatrix} - 3\begin{bmatrix}5\\-3\\6\end{bmatrix} =$$

$$\begin{bmatrix}12\\-8\\14\end{bmatrix} + \begin{bmatrix}-15\\9\\-18\end{bmatrix} =$$

$$\begin{bmatrix}-3\\1\\-4\end{bmatrix}$$

$$\begin{bmatrix}8 & 3 & -4\\5 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}8\\5\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}-4\\2\end{bmatrix} =$$

$$\begin{bmatrix}7\\8\end{bmatrix}$$

Exercise 4. Do the following vectors span $\mathbb{R}^3$?

$$\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$$

-2 8 -5

Theorem 4 on pg. 37 of the book says the columns of a matrix A span $\mathbb{R}^m$ iff A has a pivot position in every row.
Let's find the pivot positions of the matrix $\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix}$

Rearranging the order of the rows gets us the following row-echelon form.

$$\begin{bmatrix} -2 & 8 & 5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

I circled the pivot positions, as you can see, there is a pivot position in every row. So, by theorem 4, there do span $\mathbb{R}^3$.

Exercise 5. Let A be a $3{\times}2$ matrix. Explain why the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. (i.e. you can always find a $\mathbf{b}$ such that the equation is inconsistent) Generalize your argument to the case of an arbitrary A with more rows than columns.

Proof. Suppose we have an $r \times c$ matrix where r is bigger than c. Note that since there can be at most one pivot per column, the number of pivots is less than or equal to c. Since the number of rows is greater than c, M must have at least one row without a pivot. If a row does not have a pivot, it is all zeros. We now consider the augmented matrix. Since the echelon form of the coefficent matrix has a row of all zeros, we are able to choose a $\mathbf{b}$ so that the last row in the augmented matrix is of the form $[0, 0, \dots, 0 \;\; c]$ where c is nonzero, making the system inconsistent. $\square$

Exercise 6. Describe all solutions of $A\mathbf{x} = \mathbf{0}$, for the following matrices. Express your answers in parametric vector form.

$$A = \begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & 0 & -8 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & -6 \end{bmatrix}$$

We row-reduce the augmented matrix:

$$\begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

x_2, x_3, x_4 are free variables.

Writing x_1 in terms of these, we get

$x_1 = -3x_2 + 4x_4$.

In parametric vector form, we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \xrightarrow{2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix}$$

x_3, x_4 are free variables

Writing x_1, x_2 in terms of these, we get:

$x_1 = 5x_3 + 7x_4$

$x_2 = -2x_3 + 6x_4$.

In parametric vector form, we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 7. Describe the solutions of the system given by the following augmented matrix. Express your answer in parametric vector form.

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix}$$

We first row-reduce the matrix:

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \xrightarrow{4R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{bmatrix} \xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2}$$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-3R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in reduced row echelon form.

I circled the pivot positions, we see that x_3 is a free variable.

We can write the solutions as

$x_1 = 5x_3 + 2$

$x_2 = -2x_3 - 1$.

$x_3 = x_3$.

In parametric vector form, this is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

Exercise 8. Determine if each set of vectors is linearly independent.

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 343 \\ 454 \\ 55 \\ -45 \\ 67 \end{bmatrix}$$

(a). We check to see whether there are nontrivial solutions to $x_1\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The augmented matrix is

$$\begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & -8 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rearrange rows}} \begin{bmatrix} 2 & -8 & 1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

There are no free variables, so only the trivial solution exists. So these vectors are $\boxed{\text{linearly independent}}$

(b) This set is $\boxed{\text{linearly dependent,}}$ since the second vector is a scalar multiple of the first.

(c). There exists a nontrivial linear combination that sums to 0:

$$1\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + 0\begin{bmatrix} 343 \\ 454 \\ 55 \\ -45 \\ 67 \end{bmatrix} = 0$$

So the set is $\boxed{\text{linearly dependent.}}$

Exercise 9. Determine the possible row echelon forms of a 2×2 matrix with linearly dependent columns.

The possible row-echelon forms of a 2×2 matrix are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & \star \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & \star \\ 0 & \blacksquare \end{bmatrix}.$$

If we view these as coefficient matrices, A of the augmented matrix $\begin{bmatrix} A & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix}$, $\begin{bmatrix} \blacksquare & \star & 0 \\ 0 & \blacksquare & 0 \end{bmatrix}$ is the only one that has no free variables.

So $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & \star \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$ are the possible row echelon forms for a 2×2 matrix with linearly dependent columns.

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