Worksheet 2

Sections 2017 and 219
MATH 54

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Exercise 1. The following three matrices are already in row echelon form. Which represent a consistent system of equations? How many solutions does each system have?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

consistent, one solution

$0 \neq 4$ so this is inconsistent.

consistent, ∞ many solutions, since x_2 is a free variable.

Exercise 2. Put the following in row echelon form.

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix} \qquad \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$$

Then put each into reduced echelon form and describe the solution set.

We first put into row echelon form:

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & -1 & 0 & -4 \end{bmatrix}$$

We now make ~~one~~ two small adjustments to put it into reduced row echelon form:

$$\xrightarrow{-1R_2} \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 1 & 0 & 4 \end{bmatrix} \xrightarrow{-4R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \end{bmatrix}.$$

The solution is: $x_1 = -9$, $x_2 = 4$, x_3 is free.

We first put into row echelon form:

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \xrightarrow{R_1+R_3 \to R_3} \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix} \xrightarrow{4R_2+R_3 \to R_3}$$

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in row echelon form. It is also in reduced row echelon form. The solution is:

x_2, x_4 are free,
$x_1 = 7x_2 - 6x_4 + 5$
$x_3 = 2x_4 - 3$.

Note: I just did this in one possible way, there are many :)

Exercise 3. Describe the possible echelon forms of a nonzero 3×2 matrix. Use the symbols $\square$, $*$, and 0, where $\square$ means a nonzero number and $*$ means any number.

$$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Exercise 4. Find h, k such that the system below has: (a) no solutions, (b) a unique solution, and (c) infinitely many solutions.

$$x + hy = 2$$
$$4x + 8y = k$$

The augmented matrix is $\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix}$

One possible row echelon form is

$\begin{bmatrix} 1 & h & 2 \\ 0 & 8\text{-}4h & k\text{-}8 \end{bmatrix}$

(a). This is inconsistent if and only if the last row is of the form $[0, 0, a]$, $a \neq 0$.

So we want $8-4h=0$, $k-8 \neq 0$

So this is inconsistent ~~if and only if~~

$\boxed{h=2, k \neq 8.}$

(b) There is a unique solution if and only if there is a pivot in the second column. For this to happen, $8-4h \neq 0$.

$\boxed{\text{So } h \neq 2.}$

(c) Here, there are ∞ many solutions if and only if x_2 is a free variable, and the system is consistent. ie. the last row is all 0's.

So $8-4h=0$ and $k-8=0$

which simplifies to

$\boxed{h=2, k=8}$

Exercise 5. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

Note that the coef. matrix is the augmented matrix with the last column removed. If there is a pivot in every row of the coeff matrix, then the augmented matrix in echelon form does not have $[\underbrace{0\ 0\ \ldots\ 0}\ \ b]$ (nonzero) as a row.

A pivot in the coeff matrix implies one of these entries is nonzero.

Exercise 6. A system of linear equations with more equations than unknowns is sometimes called *overdetermined.* Can such a system be consistent? Illustrate your answer with a specific system of 3 equations and 2 unknowns. (It may be helpful to draw a picture in the plane!)

Yes! Any system of 3 lines that intersects at one point works!

For example:

$$\begin{cases} x=0 \\ y=0 \\ x=y \end{cases}$$ is consistent, $x=0, y=0$ is a solution.