Worksheet 25

Sections 207 and 219
MATH 54

May 2, 2019

Exercise 1. Find a general solution for the given boundary value problems!

(a) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < \pi$, $t > 0$
$u(0,t) = 0$, $u(\pi, t) = 3\pi$

(b) $\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} + x$, $0 < x < \pi$, $t > 0$
$u(0,t) = u(\pi,t) = 0$

(a). Here, we have an inhomogenity in the boundary conditions. We begin by proposing a solution of the form

$u(x,t) = w(x,t) + v(x)$,

where $w(x,t)$ is a solution to the case with homogenous boundary conditions:

$w(0,t) = 0, w(\pi,t) = 0.$

From the book's discussion in section 10.1, we know that this solution is

$$w(x,t) = \sum_{n=1}^{\infty} c_1 e^{-n^2 t} \sin(nx).$$

We now consider $v(x)$:

When $t \to \infty$, $w(x,t)$ goes to 0, so $u(x,t) = v(x)$ (we call $v(x)$ the steady state solution since it doesn't depend on time).

Plugging this into $\frac{\partial u}{\partial t} = \frac{\partial^2 v}{\partial x^2}$, we get $v''(x) = 0$. Our bondary conditions are $v(0) = 0$, $v(\pi) = 3\pi$.

Our aux equation is $r^2 = 0$, so $r = 0$. our general sontion is $v(x) = Ax + B$. Since $V(0) = 0$,

$V(0) = A(0) + B = B = 0.$

So $v(x) = Ax$. Pluggin in our other boundary condition, we get $v(\pi) = A\pi = 3\pi$. So $A = 3$

So $v(x) = 3x$.

So our general sontion is

$$u(x,t) = \left(\sum_{n=1}^{\infty} c_1 e^{-n^2 t} \sin(nx)\right) + 3x$$

(b) Again, we guess the solution is of the form $u(x,t) = w(x,t) + v(x)$, where $w(x,t)$ is the solution of our system without the extra x added on:

$$\frac{\partial w}{\partial t} = 3\frac{\partial^2 w}{\partial x^2}, \quad w(0,t) = w(\pi,t) = 0$$

From the book's discussion we know that $w(x,t) = \sum_{n=1}^{\infty} c_1 e^{-3n^2 t} \sin(nx)$

We now consider $v(x)$. When $t \to \infty$, $u(x,t) = v(x)$. Plugging this into our original equation / boundary conditions, we get

$$v''(x) = -\frac{x}{3} \quad v(0) = v(\pi) = 0.$$

We can determine $v(x)$ by antidifferentiating twice: if

$$v''(x) = -\frac{x}{3} \text{ then } v'(x) = -\frac{x^2}{6} + C_1$$

$$\text{So } v(x) = -\frac{x^3}{18} + C_1 x + C_2.$$

Since $v(0) = 0$, $v(0) = C_2 = 0$

$$\text{So } v(x) = -\frac{x^3}{18} + C_1 x.$$

$$\text{So } v(\pi) = -\frac{\pi^3}{18} + C_1\pi = 0$$

$$\text{So } C_1 = \frac{\pi^2}{18}$$

$$\text{So } v(x) = -\frac{x^3}{18} + \frac{\pi^2}{18} x$$

So the general solution is

$$u(x,t) = -\frac{x^3}{18} + \frac{\pi^2}{18} x + \sum_{n=1}^{\infty} c_1 e^{-3n^2 t} \sin(nx)$$