Worksheet 22

Sections 207 and 219
MATH 54

April 23, 2019

Exercise 1. Use the method of separation of variables to write the following partial differential equation as a system of 2 ordinary differential equations (using some parameter λ):

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

where u is a function of x and t.

There is a blurb on this at the beginning of section 10.2

Exercise 2. Determine all solutions, if any, to the following boundary value problem:

$$y'' + 9y = 0, \; 0 < x < \pi$$

$$y(0) = 0, \; y'(\pi) = -6$$

We first find the general solution:
The auxiliary equation is $r^2+9=0$,
so $r = \pm 3i$. So
$y = C_1 \sin(3x) + C_2 \cos(3x)$.

We now plug in our initial condition $y(0)=0$:

$y(0) = C_1 \sin(0) + C_2 \cos(0)$

$= C_2 = 0$. So $C_2 = 0$.

So $y(x) = C_1 \sin(3x)$.

We now ~~want~~ to consider our second boundary condition.
First we note that
$y'(x) = 3C_1 \cos(3x)$.

so $y'(\pi) = 3C_1(\cos 3\pi) = -3C_1 = -6$.

So $C_1 = 2$.

Thus:

$y(x) = 2\sin(3x)$.

Exercise 3. Find all values of λ for which the given problem has a nontrivial solution. Then find the nontrivial solution.

$$y'' - 2y' + \lambda y = 0,\ 0 < x < \pi$$
$$y(0) = 0,\ y(\pi) = 0$$

First, the auxiliary equation is: $r^2 - 2r + \lambda = 0$, which has roots

$r = \frac{2 \pm \sqrt{4-4\lambda}}{2} = 1 \pm \sqrt{1-\lambda}$.

The general solution looks different depending on whether or not the roots are distinct real, repeated real, or imaginary. So we break this up into 3 cases.

Case 1: 2 distinct real roots, r_1, r_2. (Note that this happens when $\lambda < 1$).

Then $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. Plugging in $y(0)=0$, we get

$y(0) = c_1 + c_2 = 0$ so $c_2 = -c_1$.

So $y(x) = c_1 e^{r_1 x} - c_1 e^{r_2 x} = c_1(e^{r_1 x} - e^{r_2 x})$.

Plugging in $y(\pi)=0$, we get

$y(\pi) = c_1 \underbrace{(e^{r_1\pi} - e^{r_2\pi})}_{\text{This is not 0 since } r_1 \neq r_2} = 0$.

so $c_1 = 0$. since $c_1 = -c_2 = 0$, this case has only the trivial solution.

Case 2: 1 distinct root. This happens when $\lambda = 1$, giving us a root of $r = 1 \pm \sqrt{0} = 1$.

So $y(x) = c_1 e^x + c_2 x e^x$. Plugging in $y(0)=0$, we get $y(0) = c_1 + 0 = c_1 = 0$.

So $y(x) = c_2 x e^x$. Plugging in $y(\pi) = 0$, we get $y(\pi) = c_2 \pi e^\pi = 0$. since $\pi e^\pi \neq 0$, c_2 must be 0. So again there are no nontrivial solutions.

Case 3: 2 complex roots. This happens when $\lambda > 1$. So

$r = 1 \pm \sqrt{1-\lambda} = 1 \pm i\sqrt{\lambda - 1}$. So.

$y(x) = c_1 e^x \sin(\sqrt{\lambda-1}\,x) + c_2 e^x \cos(\sqrt{\lambda - 1}\,x)$.

$y(0) = c_1 \sin 0 + c_2 \cos(0) = c_2 = 0$.

So $y(x) = c_1 e^x \sin(\sqrt{\lambda-1}\,x)$. We now use our other boundary condition:

$y(\pi) = c_1 e^\pi \sin(\sqrt{\lambda-1}\,\pi) = 0$. We have nontrivial solutions iff it is possible for $c_1 \neq 0$. This happens precisely when $\sin(\sqrt{\lambda-1}\,\pi) = 0$, which happens when $\sqrt{\lambda-1}$ is an integer, n. so

$\sqrt{\lambda - 1} = n \Rightarrow \lambda = n^2 + 1$.

So our eigenvalues are $\lambda_n = n_2 + 1$. The corresponding eigenfunctions are: $y_n(x) = c_1 e^x \sin(\sqrt{(n^2+1)-1}\,x) =$

$c_1 e^x \sin(nx)$.