Worksheet 21

Sections 207 and 219
MATH 54

April 18, 2019

Exercise 1. Consider $\mathbf{x}'(t) = A\mathbf{x}(t)$, $t \geq 0$, where $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

(a) Show that the eigenvalues are -1,-3 and $\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are the corresponding eigenvectors.

(b) Sketch the tractory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u_1}$.

(c) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = -\mathbf{u_2}$.

(d) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u_1} - \mathbf{u_2}$.

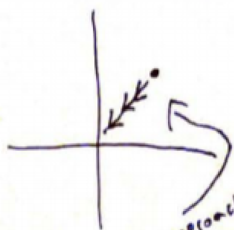
(a). $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-2&1\\1&-2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix} = -1\begin{bmatrix}1\\1\end{bmatrix}$. So $\begin{bmatrix}1\\1\end{bmatrix}$ is indeed an eigenvec with corr. eigenval -1.

$A\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-2&1\\1&-2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-3\\3\end{bmatrix} = -3\begin{bmatrix}1\\-1\end{bmatrix}$. So $\begin{bmatrix}1\\-1\end{bmatrix}$ is indeed an eigenvec with corr eigenval -3.

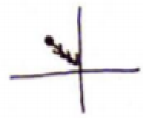
(b). The solution corr to this IVP is

$\vec{x}(t) = e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$.

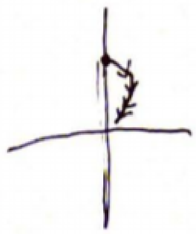
As t gets bigger, $\vec{x}(t)$ gets closer to 0. every point on the trajectory is a scalar mult of $\begin{bmatrix}1\\1\end{bmatrix}$.



(c) Solution is $\vec{x}(t) = -e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix} = e^{-3t}\begin{bmatrix}-1\\1\end{bmatrix}$.



(d). Solution is $\vec{x}(t) = e^{-t}\begin{bmatrix}1\\1\end{bmatrix} - e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}e^{-t} - e^{-3t}\\e^{-t} + e^{-3t}\end{bmatrix}$.



Facts about this:

- Approaches origin as $t \to \infty$.
- both coords are always positive.
- First coord is always less than second. so above line $y = x$.

Exercise 2. Give a general solution to:

$$\mathbf{x}'(t) = \begin{bmatrix}6 & -3\\2 & 1\end{bmatrix}$$

Eigenvalues are solutions to: $(6-\lambda)(1-\lambda)+6 = \lambda^2 - 7\lambda + 12 = (\lambda-3)(\lambda-4) = 0$

So $\lambda = 3, 4$.

We now find eigenvecs.

$\lambda = 3$: $\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right]$ $-x_1 + x_2 = 0$, $x_1 = x_2$.

So eigenvects are of form $c_1\begin{bmatrix}1\\1\end{bmatrix}$

$\lambda = 4$ $\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right]$ So $x_1 = 3x_2$.

So eigenvecs are of form

$\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\begin{bmatrix}3\\1\end{bmatrix}$.

Putting everything together, our general solution is:

$\vec{x}(t) = c_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}3\\1\end{bmatrix}$.

We are assuming $a \neq 0$.

Exercise 3. Use the substitutions $x_1 = y$, $x_2 = y'$ to convert $ay'' + by' + cy = 0$ into a normal system. Show that the characteristic equation of the system is the same as the aux equation for the original equation.

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Let $x_1 = y$, $x_2 = y'$. Then:

$$\vec{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{b}{a}x_2 - \frac{c}{a}x_1 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 0 \\ -c/a \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -b/a \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix} \vec{x}.$$ So

our normal system is $\vec{x}' = A\vec{x}$ where

$$A = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}$$ The char equation

of the system is the same as the char equation of the matrix:

$$(-\lambda)(-b/a - \lambda) + c/a = 0$$

This rearranges to

$$b/a\,\lambda + \lambda^2 + c/a = \lambda^2 + b/a\,\lambda + c/a = 0.$$

Multiplying both sides by a, we get

$$a\lambda^2 + b\lambda + c = 0$$

which is indeed the aux equation of the equation

$$ay'' + by' + cy = 0$$

Exercise 4. Find a general solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ for the given matrix A. You can use the fact that the eigenvalues of A are 2, $2+i$, and $2-i$.

$$A = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}$$

I used Wolfram Alpha to find that the eigenvalues/vectors are:

$\lambda_1 = 2$, $v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ Using section 7.5, This corresponds to a solution of

$$e^{2t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

The remaining two eigenvalues are $2 \pm i$ with cor. eigenvectors $\begin{bmatrix} 5 \\ -2 \pm i \\ 5 \end{bmatrix}$.

Recall that a formula for 2 lin. ind solutions is:

$$e^{\alpha t}\cos\beta t\,\vec{a} - e^{\alpha t}\sin\beta t\,\vec{b}$$

and

$$e^{\alpha t}\sin\beta t\,\vec{a} + e^{\alpha t}\cos\beta t\,\vec{b}$$

Here, $\alpha = 2$, $\beta = 1$, $\vec{a} = \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ So our two solutions are

$$e^{2t}\cos t\begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix} - e^{2t}\sin t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e^{2t}\begin{bmatrix} 5\cos t \\ -2\cos t - \sin t \\ 5\cos t \end{bmatrix}$$

and

$$e^{2t}\sin t\begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix} + e^{2t}\cos t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e^{2t}\begin{bmatrix} 5\sin t \\ -2\sin t + \cos t \\ 5\sin t \end{bmatrix},$$

Putting everything together, our general solution is:

$$x(t) = C_1 e^{2t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + C_2 e^{2t}\begin{bmatrix} 5\cos t \\ -2\cos t - \sin t \\ 5\cos t \end{bmatrix} + C_3 e^{2t}\begin{bmatrix} 5\sin t \\ -2\sin t + \cos t \\ 5\sin t \end{bmatrix}.$$