Worksheet 24

Sections 306 and 310
MATH 54

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Exercise 1. Write the given system in normal matrix form: $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

$$x'(t) = x + y + z$$
$$y'(t) = 2x - y + 3z$$
$$z'(t) = x + 5z + e^{5t}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^{5t} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 0 & 5 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ e^{5t} \end{bmatrix}}_{f.}$$

Similar in flavor to putting solutions into parametric vector form.

Exercise 2. Rewrite the given equation as a first order system in normal form:

$$y''' - y' + y = \cos(t)$$

We first set up some extra functions. We let $x_1 = y$, $x_2 = y'$, $x_3 = y''$.

We now want to find expressions for x_1', x_2', and x_3' in terms of x_1, x_2, x_3.

$$x_1' = y' = x_2$$
$$x_2' = (y')' = y'' = x_3$$
$$x_3' = (y'')' = y''' = \cos(t) - y + y' = \cos(t) - x_1 + x_2$$

Now, we just do what we did in exercise 1 to write this system in normal form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = x_1\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos(t) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos(t) \end{bmatrix}$$

Exercise 3. Determine whether the given vector functions are linearly independent of linearly dependent on $(-\infty, \infty)$.

(a) $e^t \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $e^t \begin{bmatrix} -3 \\ -15 \end{bmatrix}$

(b) $\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$, $\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$

(a). These are linearly dependent, since they are scalar multiples of each other.

(b). These are linearly independent. To prove this, we first compute the wronskian.

$$W\left[\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}\right](t) = \begin{vmatrix} \sin t & \sin(2t) \\ \cos t & \cos(2t) \end{vmatrix} = \sin t \cos(2t) - \cos t \sin(2t).$$

We show this is nonzero when $t = \frac{\pi}{2}$. $W(t) = \sin(\frac{\pi}{2})\cos(\pi) - \cos(\frac{\pi}{2})\sin(\pi) = 1(-1) - 0 \cdot 0 = -1 \neq 0$.

There's a thm in John Lott's lecture notes that says if $\vec{y}_1, \ldots \vec{y}_n$ are LD, then W will always be 0. So since we found a value of t for which $W(t) \neq 0$, y_1, y_2 can't be LD. So they must be LI.

Exercise 4. The given vector functions are solutions to a system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Do they form a fundamental set? If so, find a fundamental matrix and five a general solution.

$$\mathbf{x_1}(t) = e^{-t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \mathbf{x_2}(t) = e^{4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Suppose A is an $n \times n$ matrix. Then, a set of vector functions is a fundamental set if:

- each function is a solution to the system
- we have n vector functions in our set.
- The set is LI. (linearly independent)

We can see that the first two conditions hold in this case. It remains to check that $\vec{x}_1, \vec{x}_2$ are LI.

Again we use the Wronskian.

$$W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} 3e^{-t} & e^{4t} \\ 2e^{-t} & -e^{4t} \end{vmatrix} = -5e^{3t}.$$

This function is not identically 0 (in fact it is never 0) so by the reasoning in 3(b), $\vec{x}_1, \vec{x}_2$ are LI. Thus, they form a fundamental set.

Fun. Matrix $X(t) = \begin{bmatrix} 3e^{-t} & e^{4t} \\ 2e^{-t} & -e^{4t} \end{bmatrix}$

Gen Solution $\vec{x}(t) = c_1 \begin{bmatrix} 3e^{-t} \\ 2e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{4t} \\ -e^{4t} \end{bmatrix} = \begin{bmatrix} 3e^{-t} & e^{4t} \\ 2e^{-t} & -e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

either of th[…] forms is ok

Exercise 5. Let $X(t)$ be the fundamental matrix for the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Show that $\mathbf{x}(t) = X(t)X^{-1}(t_0)\mathbf{x_0}$ is the solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$, and $\mathbf{x}(t_0) = \mathbf{x_0}$.

In order to show that the given function solves the initial value problem, we have to show two things:

(a). $\vec{x}(t)$ is indeed a solution to $\vec{x}' = A\vec{x}$

(b). $\vec{x}(t)$ satisfies the initial condition $\vec{x}(t_0) = \vec{x}_0$.

(a). Since $X(t)$ is a fundamental matrix for ~~solution~~ the homogeneous system, every solution is of the form $X(t)\vec{c}$, where $\vec{c}$ is an appropriately-sized vector of constants. Furthermore, every function of this form is a solution of $\vec{x}' = A\vec{x}$.

Consider $\vec{x}(t) = X(t)\underbrace{X^{-1}(t_0)\vec{x}_0}_{\text{this multiplies out to be a vector of constants}}$

So by the above reasoning, $\vec{x}(t)$ is indeed a solution to $\vec{x}' = A\vec{x}$.

(b) Let's plug t_0 into $\vec{x}'(t)$.

$\vec{x}(t_0) = X(t_0)X^{-1}(t_0)\vec{x}_0 = I\vec{x}_0 = \vec{x}_0$ as desired.

So $\vec{x}(t)$ does indeed satisfy the initial condition.