Worksheet 1

Sections 207 and 219
MATH 54

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Exercise 1. For each augmented matrix, write a corresponding system of linear equations. Can you tell (without doing any calculations) that one of these systems has no solutions?

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 3 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\begin{cases} x = 2 \\ y = 3 \\ z = 0 \end{cases} \qquad \begin{cases} y = 1 \\ x + 3y + 2z = 2 \\ z = 3 \end{cases} \qquad \begin{cases} x = 4 \\ 3z = 5 \\ 0 = 6 \end{cases}$$

This system has no solutions, since the last equation is false.

Exercise 2. Write each of the following systems as an augmented matrix. Then, solve each system.

1. • $x_1 + 5x_2 = 3, \quad x_1 - x_2 = -3$
2. • $x - 2y = 4, \quad -3x + 6y = -12$
3. • $x - 2y = 4, \quad -3x + 6y = 5$
4. • $x_1 - 3x_2 = 5, \quad -x_1 + x_2 + 5x_3 = 2, \quad x_2 + x_3 = 0$

1 $\begin{bmatrix} 1 & 5 & 3 \\ 1 & -1 & -3 \end{bmatrix} \xrightarrow{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 5 & 3 \\ 0 & -6 & -6 \end{bmatrix}$

$\xrightarrow{-\frac{1}{6}R_2} \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-5R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

So $x_1 = -2$, $x_2 = 1$.

2 $\begin{bmatrix} 1 & -2 & 4 \\ -3 & 6 & -12 \end{bmatrix} \xrightarrow{3R_1+R_2 \to R_1} \begin{bmatrix} 1 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

So the solution set is all x, y that satisfy $x - 2y = 4$.
One way of writing this is $(2y+4, y)$.

3 $\begin{bmatrix} 1 & -2 & 4 \\ -3 & 6 & 5 \end{bmatrix} \xrightarrow{3R_1+R_2 \to R_2}$

$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 0 & 13 \end{bmatrix}$ Since $0 \neq 13$, there are no solutions.

4 $\begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_1+R_2 \to R_2}$

$\begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$

$\begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{bmatrix} \xrightarrow[2R_2+R_3 \to R_3]{3R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{bmatrix}$

$\xrightarrow{\frac{1}{7}R_3} \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[-3R_3+R_1 \to R_1]{-R_3+R_2 \to R_2}$

$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

So $x_1 = 2$, $x_2 = -1$, $x_3 = 1$.

Exercise 3. If possible, compute each of $3C - E$, CB, EB. If any of these computations are impossible, briefly explain why.

① ② ③

$$B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

① Impossible since 3C is 2×2 and E is 2×1. To add/subtract matrices, the dimensions have to match.

②
$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 7+2 & -5-8 & 1-6 \\ -14+1 & 10-4 & -2-3 \end{bmatrix} = \begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}$$

③ Impossible. E is 2×1, B is 2×3. The number of columns of E would have to match the number of columns of B for the multiplication to be possible.

Exercise 4. If a matrix B is 5×3 and the product AB is 2×3, what is the size of A? (For an $m \times n$ matrix, m is the number of rows and n is the number of columns.)

We know that A is m×n, B is 5×3, and AB is 2×3. In order for the multiplication to even be possible, n has to match with the number of rows of B, so n=5. Also, m ~~[illegible]~~ matches with the number of rows of AB, so m=2. So A is 2×5.

Exercise 5. (bonus problem!) Show that the following equation holds:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} a \\ d \\ g \end{bmatrix} + x_2 \begin{bmatrix} b \\ e \\ h \end{bmatrix} + x_3 \begin{bmatrix} c \\ f \\ i \end{bmatrix}$$

The fact that multiplying a matrix by a vector gives a weighted sum of the columns of the matrix will be useful later! Don't worry too much if this problem doesn't make too much sense right now.

We will work out the multiplication and show that the left hand side is indeed equal to the right hand side.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 + cx_3 \\ dx_1 + ex_2 + fx_3 \\ gx_1 + hx_2 + ix_3 \end{bmatrix} = \begin{bmatrix} ax_1 \\ dx_1 \\ gx_1 \end{bmatrix} + \begin{bmatrix} bx_2 \\ ex_2 \\ hx_2 \end{bmatrix} + \begin{bmatrix} cx_3 \\ fx_3 \\ ix_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a \\ d \\ g \end{bmatrix} + x_2 \begin{bmatrix} b \\ e \\ h \end{bmatrix} + x_3 \begin{bmatrix} c \\ f \\ i \end{bmatrix}$$ as desired.