Worksheet 22

Sections 306 and 310
MATH 54

Nov 6, 2018

Exercise 1. Can the method of undetermined coefficients be used to find a particular solution for the following?

(a) $y'' + 2y' - y = t^{-1}e^t$ no! t^{-1} is not a polynomial.

(b) $y'' + 2y' - y = te^{-t}$ yes!

(c) $2y'' - 3y = 4t\sin^2(t) + 4t\cos^2(t)$ yes! $4t\sin^2(t) + 4t\cos^2(t) = 4t(\sin^2 t + \cos^2 t) = 4t$.

Exercise 2. Find a particular solution for each of the following:

(a) $y'' + 4y = 8\sin(2t)$

(b) $y'' - 5y' + 6y = te^t$

(a). Our guess is

$y_p = t^s(A\sin(2t) + B\cos(2t))$

To determine what s is, we solve the aux. equation of the homogenous DE:

$r^2 + 4 = 0 \Rightarrow r = \pm 2i$.

Since this "matches" with our guess of $A\sin(2t) + B\cos(2t)$, our particular solution needs an extra factor of t.

So $s = 1$.

Thus, $y_p = t(A\sin(2t) + B\cos(2t))$.

We are now ready to find A, B.

$y_p'' = 4\cos(2t)(A - Bt) - 4\sin(2t)(At + B)$.

Plugging these in, we get

$y_p'' + 4y = 4\cos(2t)(A - \cancel{Bt} + \cancel{Bt}) - 4\sin(2t)(At + B + At)$
$= 8\sin(2t)$

setting coefs equal to each other, we get:

$4A = 0$
$8A = 0$
$-4B = 8$. So $A = 0$, $B = -2$

So $y_p = -2t\cos(2t)$.

(b) Our guess is $y_p = t^s(At + B)e^t$

To determine what s is, we solve the aux equation

$r^2 - 5r + 6 = (r-2)(r-3) = 0 \Rightarrow r = 2, 3$.

neither of the roots match with the 1 (from e^{1t})

So $s = 0$ and $y_p = (At + B)e^t$.

So $y_p' = Ae^t + Ate^t + Be^t$

$y_p'' = 2Ae^t + Ate^t + Be^t$

Plugging these in and rearranging terms, we get

$2Ate^t + (2B - 3A)e^t = te^t$.

So $2A = 1$, $2B - 3A = 0$ $\Rightarrow$ $A = \frac{1}{2}$, $B = \frac{3}{4}$.

So $y_p = \frac{1}{2}te^t + \frac{3}{4}e^t$.

Exercise 3. Find the form of a particular solution of the following equation, but do not evaluate the coefficients.

$$y'' - y' - 12y = 2t^6 e^{-3t}$$

The form is $y_p = t^s(A + Bt + Ct^2 + Dt^3 + Et^4 + Ft^5 + Gt^6)e^{-3t}$

The only thing we have left to do is determine ~~what~~ what s is.

To this, we look at the roots of the aux eq. $r^2 - r - 12 = (r+3)(r-4) = 0$

So $r = -3, 4$. Since $r = -3$ matches up with the -3 in e^{-3t}

$s = 1$.

So $y_p = t(A + Bt + Ct^2 + Dt^3 + Et^4 + Ft^5 + Gt^6)e^{-3t}$.

Luckily, we just want the form, and do not have to determine the 7 undetermined coefficents.

Exercise 4. Find a general solution to the following differential equation:

$$y'' + 4y = \sin(t) - \cos(t)$$

Let's first find the general solution to the homog. equation $y'' + 4y = 0$. The aux equation is $r^2 + 4y = 0$ which has roots $r = \pm 2i$.

So $y(t) = c_1 \sin(2t) + c_2 \cos(2t)$.

We now we und. coeff to find a particular solution.

The general form is

$y_p = t^s(A\sin t + B\cos t)$. Since the

This comes from $e^{0t}\sin t$

$(\pm i)$ are not roots of aux eq., $s = 0$.

So $y_p = A\sin t + B\cos t$.

$y_p' = A\cos t - B\sin t$

$y_p'' = -A\sin t - B\cos t$.

Plugging this in, we get

$-A\sin t - B\cos t + 4A\sin t + 4B\cos t = \sin t - \cos t$.

Matching up coefficients, we get.

$3A = 1 \Rightarrow A = \frac{1}{3}$

$3B = -1$, $B = -\frac{1}{3}$

So $y_p = \frac{1}{3}\sin t - \frac{1}{3}\cos t$.

So the general solution is

$y(t) = c_1 \sin(2t) + c_2 \cos(2t) + \frac{1}{3}\sin t - \frac{1}{3}\cos t$

Exercise 5. All that is know about a mysterious second-order constant-coefficient differential equation $y'' + py' + qy = g(t)$ is that $\underbrace{t^2 + 1 + e^t \cos(t)}_{y_1}$, $\underbrace{t^2 + 1 + e^t \sin(t)}_{y_2}$, and $\underbrace{t^2 + 1 + e^t \cos(t) + e^t \sin(t)}_{y_3}$ are solutions.

(a) Determine the general form of solutions to the homogeneous equation.

(b) Find a suitable choice of p, q, and $g(t)$ that enables these solutions.

(a). By the law of superposition, we can see that $y_3 - y_1$ and $y_3 - y_2$ are solutions to $y'' + py' + qy = g(t) - g(t) = 0$. So $y_3 - y_1 = e^t \sin t$ and $y_3 - y_2 = e^t \cos t$ are solutions to the homogenous equation. Since we have two linearly independent solutions to a second order homog. equation, we have enough info to write the general solution to the homog equation:

$$y(t) = c_1 e^t \sin t + c_2 e^t \cos t.$$

(b) We first find p, q such that $y'' + py' + qy = 0$ has general solution $c_1 e^t \sin t + c_2 e^t \cos t$.

This would happen if the aux equation $r^2 + pr + q = 0$ had roots $1 \pm i$. So $r^2 + pr + q = (r - (1+i))(r - (1-i)) = r^2 - (1+i)r - (1-i)r + 2$

$= r^2 - 2r + 2.$ so $p = -2$, $q = 2$.

So now we know that our original equation is $y'' - 2y' + 2y = g(t)$.

We can see from our given solutions y_1, y_2, y_3 that $t^2 + 1$ is a particular solution. We plug this into our differential equation to see what $g(t)$ is:

Scratch work

$y(t) = t^2 + 1$

$y'(t) = 2t$

$y''(t) = 2.$

$$y'' - 2y' + 2y = 2 - 4t + 2(t^2 + 1) = 2t^2 - 4t + 4.$$

So $g(t) = 2t^2 - 4t + 4.$