Worksheet 18

Sections 207 and 219
MATH 54

April 9

Exercise 1. Find a general solution to the given differential equations:

(a) $y'' - y' - 2y = 0$

(b) $y'' - 5y' + 6y = 0$

(c) $4y'' - 4y + y = 0$

(a). auxiliary equations are

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0.$

so $r = 2, -1.$

Thus, the general solution is:

$y(t) = c_1 e^{2t} + c_2 e^{-t}.$

(b). auxiliary equations are,

$r^2 - 5r + 6 = 0.$

$(r-2)(r-3) = 0$

so $r = 2, 3.$

Thus, the general solution is:

$y(t) = c_1 e^{2t} + c_2 e^{3t}$

(c) auxiliary equation is:

$4r^2 - 4r + 1 = 0.$

$(2r-1)(2r-1) = 0.$

so $r = \frac{1}{2}.$

Thus, the general solution is:

$y(t) = c_1 e^{t/2} + c_2 t e^{t/2}.$

Exercise 2. Solve the initial given value problems:

(a) $y'' + y' = 0$; $y(0) = 2$; $y'(0) = 1$

(b) $y'' - 4y' + 4y$; $y(1) = 1$; $y'(1) = 1$

Exercise 2. Solve the initial given value problems:

(a) $y'' + y' = 0$; $y(0) = 2$; $y'(0) = 1$

(b) $y'' - 4y' + 4y$; $y(1) = 1$; $y'(1) = 1$

(a). First we find the general solution:

auxiliary equation: $r^2 + r = 0$ $\Rightarrow r = 0, -1$

$r(r+1) = 0$.

So our general solution is

$y(t) = c_0 + c_1 e^{-t}$.

note, since $e^{0t} = 1$, this is just a constant.

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First we find the general solution:

Note that the repeated root makes the form of the solution different from parts (a) and (b).

aux equation:

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$.

So the general solution is:

$y(t) = c_1 e^{2t} + c_2 t e^{2t}$.

We now use the initial conditions to solve for c_1 and c_2.

$y(1) = 1$: $y(1) = 1 = c_1 e^2 + c_2 e^2$

Note: $y'(t) = 2c_1 e^{2t} + 2c_2 t e^{2t} + c_2 e^{2t}$

$y'(1) = 1$: $y'(1) = 1 = 2c_1 e^2 + 3c_2 e^2$. $\Rightarrow c_1 = 2/e^2$, $c_2 = -1/e^2$

So $y(t) = \frac{2}{e^2} e^{2t} - \frac{1}{e^2} t e^{2t}$

Exercise 3. (a) With your group, reread definition 1 in this section (restated below):

A pair of functions $y_1(t)$ and $y_2(t)$ is said to be linearly independent on an interval I if and only if neither of them is a constant multiple of the other on all of I.

(b) Are $y_1(t) = e^{3t}$ and $y_2(t) = e^{-4t}$ linearly independent?

(c) Are $y_1 = \tan^2(t) - \sec^2(t)$ and $y_2(t) = 3$ linearly independent?

(b). These are linearly independent since they are not constant multiples of each other.

We can see this by showing that their ratio

$\frac{e^{3t}}{e^{-4t}} = e^{7t}$ is not constant!

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(c). No!

remember from trig that

$\sin^2 t + \cos^2 t = 1$.

Dividing both sides by $\cos^2 t$, we get

$\tan^2 t + 1 = \sec^2 t$ which rearranges to

$\tan^2(t) - \sec^2(t) = 1$.

So even though $y_1(t)$ has a complicated formula, we see that $y_1(t) = 1$.

So $y_2(t) = 3$ is a multiple of $y_1(t)$.

So y_1, y_2 are not lin ind.

Exercise 4. (a) Explain why two functions are linearly dependent on an interval I if and only if there exist constants c_1 and c_2, not both zero, such that

$$c_1 y_1(t) + c_2 y_2(t) = 0$$

for all t in I.

(b) Discuss with your group how this connects to the idea of linear independence that we discussed in the linear algebra section of the course.

(c) Expand the defintion of linear independent functions to apply to sets of 3 or more functions.

functions.

(a). Since this is an "if and only if" statement, we have to show both direction.

$\Rightarrow$ First we show that if y_1, y_2 are lin dep on I, then there exist c_1, c_2 not both 0 such that $c_1 y_1(t) + c_2 y_2(t) = 0$.

If y_1, y_2 are lin dep, then $y_1(t) = c y_2(t)$ for some constant c. This can be rearranged to $y_1(t) - c y_2(t) = 0$, which satisfys the condition if you let $c_1 = 1$, $c_2 = -c$.

$\Leftarrow$ We now show that if there exist c_1, c_2, not both 0, such that $c_1 y_1(t) + c_2 y_2(t) = 0$, then y_1, y_2 are lin dep.

Case 1: $c_1 \neq 0$. We can divide both sides by c_1 and rearrange to get $y_1(t) = \frac{-c_2}{c_1} y_2(t)$, satisfying the definition of lin dep.

Case 2: $c_2 \neq 0$. Similar to case 1.

(c) $y_1, \dots y_n$ are lin ind on interval I if

$c_1 y_1(t) + \dots + c_n y_n(t) = 0$ on all of I only when

$c_1 = c_2 = \dots = c_n = 0$

(this is the same idea as for lin ind vectors).

Exercise 5. We are starting a new section of this course, so there may be vocabulary that you have not seen before. In the context of this chapter, make sure you know what each of the words/phrases means in your own words:

(a) an interval I

(b) general solution

(c) initial value problem

(d) uniqueness

(a): An interval is the set of ~~[illegible]~~ real numbers between two endpoints a, b.

(b) A general solution tells you the form of all possible solutions to a differential equation.

(c) An initial value problem asks to find ~~the~~ the solution of a ~~dif. eq~~ that satisfies certain values for $y, y', \ldots$ (depending on order of equation) for a specific point.

(d) A solution to ~~[illegible]~~ a set of equations/initial value conditions is unique if it is the only one that satisfies them.

Exercise 6. Find a general solution to the given differential equations:

(a) $y'' + y = 0$

(b) $y'' - 10y' + 26y = 0$

(c) $y'' - 4y' + 7y = 0$

(a). The auxiliary equation is $r^2+1=0$, which has solutions $\pm i$

So the general solution is $y(t)=c_1e^{0t}\sin t + c_2e^{0t}\cos t = c_1\sin t + c_2\cos t$.

(b). The auxiliary equation is $r^2-10r+26=0$. Using quadratic formula, we get $r=\dfrac{10\pm\sqrt{100-104}}{2}=5\pm i$

so $y(t)=c_1e^{5t}\sin t + c_2e^{5t}\cos t$.

(c) The auxilary equation is $r^2-4r+7=0$ Using quad formula, we get $r=\dfrac{4\pm\sqrt{-12}}{2}=2\pm\sqrt{3}i$

So $y(x)=c_1e^{2x}\sin(\sqrt{3}x)+c_2e^{2x}\cos(\sqrt{3}x)$

Exercise 7. Find a general solution to the following higher-order equation:

$$y''' - y'' + y' + 3y = 0$$

The auxiliary equation is $r^3-r^2+r+3=0$

From using guess and check, we see that $r=-1$. So $(r+1)$ is a factor. You can use polynomial long devision to finish factoring!

$(r+1)(r^2-2r+3)=0$. So the roots are $r=-1$, $1\pm\sqrt{2}i$

so $y(t)=c_1e^{-t}+c_2e^{t}\sin(\sqrt{2}t)+c_3e^{t}\cos(\sqrt{2}t)$.

Exercise 8. Prove the sum of angles formula for the sine function by following these steps. Let x be a fixed constant.

(a) Let $f(t) = \sin(x + t)$. Show that $f''(t) + f(t) = 0$, $f(0) = \sin x$, and $f'(0) = \cos(x)$.

(b) Use the auxiliary techniqe to solve the initial value problem $y'' + y = 0$, $y(0) = \sin(x)$, and $y'(0) = \cos(x)$.

(c) By uniqueness, the solution in part (b) is the same as $f(t)$ from part (a). Write this equality, this should be the standard sum of angles formula for sin(x+t).

(a). Before verifying these things, we compute the first three derivative, with respect to t (ie we treat x as a constant)

$f'(t) = \cos(x+t)$, $f''(t) = -\sin(x+t)$.

So indeed, $f''(t) + f(t) = -\sin(x+t) + \sin(x+t) = 0$.

$f(0) = \sin(x+0) = \sin(x)$. as desired.

$f'(0) = \cos(x+0) = \cos(x)$

(b). The aux equation is $r^2+1=0 \Rightarrow r = \pm i$.

So $y(t) = c_1 \sin t + c_2 \cos t$, $y'(t) = c_1 \cos t - c_2 \sin t$.

We now solve initial value problem:

$y(0) = c_2 \cos 0 = c_2 = \sin x$. So $c_1 = \cos x$

$y'(0) = c_1 \cos 0 = c_1 = \cos(x)$. $c_2 = \sin x$

This is the sum of angles formula for sine.

Putting this together, $y(t) = \cos x \sin t + \sin x \cos t$.

(c). Setting $y(t) = f(t)$, we get $\cos x \sin t + \sin x \cos t = \sin(x+t)$