# Worksheet 17

Sections 207 and 219
MATH 54

April 4, 2019

**Exercise 1.** True of false! Justify!

(a) There are symmetric matrices that are not orthogonally diagonalizable.

(b) An orthogonal matrix is always orthogonally diagonalizeable.

(a). False, Thm 2 says that a matrix is orth. diag-able if and only if it is a symmetric matrix. So every sym matrix has to be orthogonally diagonalizable.

(b) False. There exist nonsymmetric orth. matrices, for example $\frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$ By thm 2, these matrices cannot be orth. diagonalizable, since they are not symmetric.

**Exercise 2.** The following matrix has eigenvalues $\lambda = -2, 7$. Orthongally diagonalize the matrix:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

(note, For this problem I just did a sketch of the procedure! If you have q's about details, email me).

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

<u>For $\lambda = 7$,</u>

You can compute that a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ Use Gram-Schmidt to turn this into an orthogonal basis: $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \right\}$ and then normalize to turn this into a orthonormal basis. $\left\{ \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix} \right\}$.

<u>For $\lambda = -2$,</u> You can compute that a basis for the eigenspace is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \right\}$ which normalizes to $\begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$.

So $A = PDP^{-1}$ where $P = \begin{bmatrix} -1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix}$, $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

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**Exercise 3.** Find an SVD for $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

This is a rough sketch:

We want to write $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = U \Sigma V^T$ where $U$, $V$ are orthogonal.

$$\underset{3\times 3}{U} \quad \underset{3\times 2}{\Sigma} \quad \underset{2\times 2}{V^T}$$

The first step is to compute $A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, and find the eigendata. The eigenvalues w. corresponding eigenvalues are:

$$\lambda_1 = 3 \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\lambda_2 = 2 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So our singular values are $\sqrt{3}, \sqrt{2}$.

We now have enough info to construct $\boxed{\Sigma : \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}}$

Note: we always order our singular values in decreasing order! We can construct $V$ by using the corresponding normalized eigenvectors as columns. (our eigenvectors already have norm 1, so we don't have to do anything. In the case in which you have a repeated sing value, you may have to orthogonalize the eigenvectors using Gram Schmidt. Again, we don't have to do that in this case)

So $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Thus $\boxed{V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

It remains to find $U$. We have a formula to find the first 2 columns:

$$\vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sigma_2} A \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$\vec{u}_3$ must be a vector that satisfies $\vec{u}_1 \cdot \vec{u}_3 = 0 \Rightarrow \frac{x_1}{\sqrt{3}} + \frac{x_2}{\sqrt{3}} + \frac{x_3}{\sqrt{3}} = 0$ and

$\vec{u}_2 \cdot \vec{u}_3 = 0 \Rightarrow \frac{x_1}{\sqrt{2}} - \frac{x_3}{\sqrt{2}} = 0.$

Any multiple of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ can be shown to be a solution of this system. Normalizing this vector, we get

$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$