Worksheet 16

Sections 207 and 219
MATH 54

π day, 2019

Exercise 1. Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthoganal basis for $\mathbb{R}^3$. Then express $\mathbf{x}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

Note: below they accidentally do not check that v_1 and v_3 are orthogonal, but this step is also necessary.

We check to make sure every pair is orthogonal.

$\vec{v}_1 \cdot \vec{v}_2 = 6 - 6 + 0 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 2 + 2 - 4 = 0$,

Since we have a set of 3 orth. vectors in $\mathbb{R}^3$, they form an orthogonal basis of $\mathbb{R}^3$.

Since the basis is orthogonal, we can use the formula in Thm 5 (page 341) to find c_1, c_2, c_3 such that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{x}$

$$c_1 = \frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{15 + 9 + 0}{9 + 9} = \frac{24}{18} = \frac{4}{3}$$

$$c_2 = \frac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = \frac{10 - 6 - 1}{4 + 4 + 1} = \frac{3}{9} = \frac{1}{3}$$

$$c_3 = \frac{\vec{x} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} = \frac{5 - 3 + 4}{1 + 1 + 16} = \frac{6}{18} = \frac{1}{3}$$

So $\vec{x} = \frac{4}{3}\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

I always forget these formulas, but the proof on page 341 helps me remember :)

Exercise 2. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in span$\{\mathbf{u}\}$ and one orthogonal to it.

$$\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

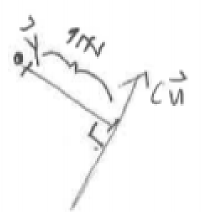
We wish to write $\vec{y} = \hat{y} + \vec{z}$, where $\hat{y}$ is in span$\{\vec{u}\}$ and $\vec{z}$ is orthogonal to it. Let W be the subspace spanned by $\vec{u}$.

Then: $\hat{y} = \text{proj}_W \vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{8-9}{16+9}\begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -4/25 \\ 3/25 \end{bmatrix}$

and $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -4/25 \\ 3/25 \end{bmatrix} = \begin{bmatrix} 54/25 \\ 72/25 \end{bmatrix}$

So $\vec{y} = \begin{bmatrix} -4/25 \\ 3/25 \end{bmatrix} + \begin{bmatrix} 54/25 \\ 72/25 \end{bmatrix}$.

From the following picture, we can see that the desired distance is $\|\vec{z}\| = \sqrt{\left(\frac{54}{25}\right)^2 + \left(\frac{72}{25}\right)^2}$.



(Sorry, I accidentally copied the wrong numbers which made the numbers gross!)

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Exercise 3. True and false! Justify your answers!

(a) If A is an $n \times n$ matrix with orthogonal columns, then it is invertible.

(b) If a set $\{\mathbf{u_1}, \ldots \mathbf{u_p}\}$ has the property that $\mathbf{u_i} \cdot \mathbf{u_j} = 0$ whenever $i \neq j$ then S is an orthnormal set.

(c) If c is not 0, then the orthogaonl projection of $\mathbf{y}$ onto a vector $\mathbf{u}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{u}$.

(a). False! For example $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ has orth. columns, but is not invertible.

· The statement "If A is an $n \times n$ matrix with orthogonal, nonzero columns, then it is invertible", is true, however.

(b) False! The $\vec{u}_i$ also have to be unit vectors for the set to be an orthonormal set.

(c) True! You can see it by drawing a picture, or by do

$$\text{proj}_{c\vec{u}} \vec{y} = \frac{c\vec{u} \cdot \vec{y}}{c\vec{u} \cdot c\vec{u}} (c\vec{u}) = \frac{\vec{u} \cdot \vec{y}}{\vec{u} \cdot \vec{u}} \vec{u} = \text{proj}_{\vec{u}} \vec{y}.$$

Warning: You can cancel out scalars in dot product computations, but you can't cancel out vectors.

Exercise 4. Let W be the subspace spanned by the $\mathbf{v}'s$ and write $\mathbf{y}$ as a sum of a vector in W and a vector orthogonal to W.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

What is the closest point in W to $\mathbf{y}$?

Again, we want to write

$\vec{y} = \hat{y} + \vec{z}$, where $\hat{y}$ is in W and $\vec{z}$ is orthogonal to W,

Using the formula in the orth. decomp. thm,

we get:

$$\hat{y} = \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \frac{\vec{y}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3}\vec{v}_3 =$$

$$\frac{1}{3}\vec{u}_1 + \frac{14}{3}\vec{u}_2 - \frac{5}{3}\vec{v}_3 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

and $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$

So $y = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$

The closest point in W to $\vec{y}$ is $\hat{y} = \text{proj}_W \vec{y} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$

Exercise 5. Find the orthogonal complement of W, where W is the span of the following two vectors:

$$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

To do this problem, use the theorem that says that the row space of A and the null space of A are orthogonal complements. Make a matrix using the given vectors as rows, and then use usual methods to find a basis of the null space.

Exercise 6. Without looking at the proof in the book, show that a set of nonzero orthogonal vectors is linearly independent.

Let $\{\vec{u}_1, \dots \vec{u}_p\}$ be an orthogonal set of nonzero vectors. Suppose there exist $c_1, \dots c_p$ such that $c_1\vec{u}_1 + \dots + c_p\vec{u}_p = 0$. In order to show $\{\vec{u}_1, \dots, \vec{u}_p\}$ is lin ind, we just have to show

$$c_1 = \dots = c_p = 0$$

We first show that $c_1 = 0$. We dot both sides by $\vec{u}_1$:

$$\vec{u}_1 \cdot (c_1 \vec{u}_1 + \dots + c_p\vec{u}_p) = \vec{u}_1 \cdot \vec{0} = 0$$

$$\Downarrow$$

$$c_1\, \vec{u}_1 \cdot \vec{u}_1 + c_2\, \vec{u}_1 \cdot \vec{u}_2 + c_3 \vec{v}_1 \cdot \vec{u}_3 + \dots + c_p \vec{u}_1 \cdot \vec{u}_p = 0.$$

Because the vectors are orthogonal, most of these terms are 0 we are left with

$$c_1 \|\vec{u}_1\|^2 = 0 \quad , \text{since } \vec{u}_1 \cdot \vec{u}_1 = \|\vec{u}_1\|^2.$$

Since $\vec{u}_1 \neq \vec{0}$, $\|\vec{u}_1\| \neq 0$. so $c_1 = 0$.

We can use the same reasoning to show that all the c_i's are 0. So $\{\vec{u}_1, \dots, \vec{u}_p\}$ are lin ind by def.