Worksheet 14

Sections 207 and 219
MATH 54

March 12, 2019

Exercise 1. (a) Find eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$ of the following matrix:

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$

(b) Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form PCP^{-1}.

Part b on next page.

(a). To find the eigenvalues, we solve the characteristic equation:

$$\begin{vmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda^2 - 8\lambda + 15) + 2 = \lambda^2 - 8\lambda + 17 = 0 \qquad \text{So } \lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm i$$

$\lambda = 4+i$: To find the eigenspace, we find the nullspace of $\begin{bmatrix} 5-(4+i) & -2 \\ 1 & 3-(4+i) \end{bmatrix}$

$\left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right]$. We know this has a nontriv solution, so since this a 2×2 matrix the second row must be some (complex) multiple of the first row. So we can solve the system using just the first equation. $((1-i)x_1 - 2(x_2) = 0$.

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{2}{1-i} \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$. So $\left\{ \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \right\}$ is a possible basis.

$\lambda = 4-i$ This time, we find the null space of $\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}$. Similarly to above, we can just use the first row: $x_1(1+i) - 2x_2 = 0$.

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{2}{1+i} \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$. So $\left\{ \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \right\}$ is a possible basis.

(b) By thm 9, on page 301. since we have an eigenvalue of $4-i$ with eigenvec $\vec{v} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$,

We can write $A = PCP^{-1}$, where $P = [\operatorname{Re}\vec{v} \;\; \operatorname{im}\vec{v}] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$,

and $C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$.

Exercise 2. The following matrix is the matrix for a composition of a rotation and a scaling. Give the angle ϕ of rotation and the scalar factor r.

$$\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$$

$$A = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$$

We pull out a factor of the magnitude of the first column:

$r = \sqrt{(-\frac{\sqrt{3}}{2})^2 + (-\frac{1}{2})^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1.$

So $A = 1 \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix} = 1 \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

where ϕ is the angle given in the following triangle

From trigonometry, we see that $\phi = -210^\circ = \frac{7\pi}{6}$ radians.

Exercise 3. True or false? Justify please! Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$.

(a) $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$

(b) dist$(\mathbf{u}, \mathbf{v})$+dist$(\mathbf{v}, \mathbf{w})$ =dist$(\mathbf{u}, \mathbf{w})$

(a) True! This is because taking the dot product is commutative.

(b) False: Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

counterexample

dist$(\vec{u}, \vec{v})$ + dist$(\vec{v}, \vec{w})$ =

$1 + 1 = 2$

but

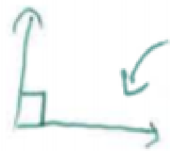
dist$(\vec{u}, \vec{w}) = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Exercise 4. Find a unit vector in the direction of the given vector. Draw a picture of what an orthogonal vector would look like.

$$\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$$

orthogonal vector would look like.

$$\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$$



orthogonal vectors form a right angle.

We divide by the magnitude of the vector, which is $\sqrt{(-6)^2+4^2+(-3)^2} = \sqrt{36+16+9} = \sqrt{61}$.

So a unit vector in the same direction is $\frac{1}{\sqrt{61}}\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

Exercise 5. True and false! Justify your answers!

(a) For any scalar c, $||c\mathbf{v}|| = c||\mathbf{v}||$.

(b) If $\mathbf{v}$ is orthoganal to every vector in a subspace W, then $\mathbf{v}$ is in $W^{\perp}$.

(c) If $||\mathbf{u}||^2 + ||\mathbf{v}||^2 = ||\mathbf{u}+\mathbf{v}||^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(d) For an $m \times n$ matrix A, vectors in nul A are orthogonal to vectors in row A.

(a) False! When c is negative, $\|c\vec{v}\|$ is positive, but $c\|v\|$ is negative.
To fix this, $\|c\vec{v}\| = |c|\,\|\vec{v}\|$ is a true statement.

(b) True! This follows from the definition of being orthogonal to a subspace.

(c) True. See Thm 2 on page 336.

(d) True! See Thm 3. on page 337.
If you are interested in a brief proof, Let $\vec{x}$ be in row A, and let $A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix}$. Then $A\vec{x} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x} \\ \vec{v}_3 \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So $\vec{x}$ is $\perp$ to all the row vectors and thus everything in row A.

These are rows.

Let me know if you have questions, :)

Exercise 6. For what values of b is the following matrix diagonalizable?

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

This is diagonalizable if and only if $b = 0$.

Case 1: $b = 0$

Then $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ which is a diagonal matrix, which is diagonalizable.

Case 2: $b \neq 0$. We know A has only one eigenvalue, a, since A is triangular and thus its eigenvalues are on the diagonal entries. We now find the eigenspace by finding the null space of $A - aI = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 0 & b & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Since $b \neq 0$, This means $x_2 = 0$ and x_1 is a free variable.

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The only eigenspace is one-dimensional, so there are not enough lin. ind eigenvectors for A to be diagonalizable.