Worksheet 13

Sections 207 and 219
MATH 54

March 7, 2019

Exercise 1. Mark each statement True or False. Justify each answer. For these problems, A, B are $n \times n$ matrices.

(a) If A, B are row equivalent, then they have the same eigenvalues.

(b) If A has n eigenvectors, A is diagonalizable.

(c) If A has n distinct eigenvalues, it is diagonalizable.

(d) If A is diagonalizable, then A has n distinct eigenvalues

(c) If A has n distinct eigenvalues, it is diagonalizable.

(a). False! row operations can change eigenvalues! For example $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(b). False. A has to have n linearly independent eigenvectors.

(c). True! There is a theorem saying that if λ_1, λ_2 are distinct eigenvalues with v_1, v_2 as corresponding eigenvectors, then $\{v_1, v_2\}$ is a linearly independent set.

(d) False! See exercise 4 for a counterexample.

Exercise 2. Let T be defined by $T(\mathbf{x}) = A\mathbf{x}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$$

By Thm. 8, if we write $A = PDP^{-1}$, and let β be the columns of P, then $[T]_\beta$ is diagonal. We first find the eigenvalues of A:

$\begin{vmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1) = 0$. So $\lambda = 3, 1$. We now find the eigenspaces:

$\lambda = 1$. We find the nullspace of $A - I1$: $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -3 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 3$ Similarly, we solve $\left[\begin{array}{cc|c} -3 & 1 & 0 \\ -3 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

So $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1/3 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. So $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \right\}$.

Exercise 3. (a) As a group, discuss why it is useful to be able to diagonalize a matrix!

(b) If possible, diagonalize the following matrix:

$$\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$$

(b) We first find eigenvalues and eigenvectors.
To find eigenvalues, we solve the characteristic equation

$\begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 15 + 1 = (\lambda - 4)^2$. So $\lambda = 4$ is the only eigenvalue. We now find the eigenspace by finding solutions to $\begin{bmatrix} 3-4 & -1 \\ 1 & 5-\lambda \end{bmatrix} \vec{y} = 0 \Rightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

So the eigenspace is $\text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$. Since there is only one eigenspace, and it is one dimensional, this matrix

Exercise 4. The eigenvalues of A are 2 and 8. Use this information to diagonalize A:

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

In order to diagonalize A, we need to find eigenspaces corresponding to each eigenvector.

$\underline{\lambda=2}$: The eigenspace is solutions to $\begin{bmatrix} 4-2 & 2 & 2 \\ 2 & 4-2 & 2 \\ 2 & 2 & 4-2 \end{bmatrix} \vec{x} = \vec{0}$

$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ So the solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\underline{\lambda=8}$: The eigenspace is solutions to $\begin{bmatrix} 4-8 & 2 & 2 \\ 2 & 4-8 & 2 \\ 2 & 2 & 4-8 \end{bmatrix} \vec{x} = \vec{0}$.

$\left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

so the solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

So we have the following linearly independent set of 3 eigenvectors

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ with 2 corresponding eigenvalues $\{2, 2, 8\}$.

So using thm 5 on pg 284, $A = PDP^{-1}$ where $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

Exercise 5. Let $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ be a bisis for a vector space V and $T : V \to \mathbb{R}^2$ be a linear transformation such that:

$$T(x_1\mathbf{b_1} + x_2\mathbf{b_2} + x_3\mathbf{b_3}) = \begin{bmatrix} 2x_1 - 4x_2 + 5x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

Find the matrix for T relative to $\mathcal{B}$ and the standard basis for $\mathbb{R}^2$.

Proof. Let $\mathcal{E}$ denote the standard basis for $\mathbb{R}^2$. The formula for the matrix ${}_{\mathcal{E}}[T]_{\mathcal{B}}$ is

$$\begin{bmatrix} [T(\mathbf{b_1})]_{\mathcal{E}} & [T(\mathbf{b_2})]_{\mathcal{E}} & [T(\mathbf{b_3})]_{\mathcal{E}} \end{bmatrix}$$

. We compute each column individually:

$T(\mathbf{b_1}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, which is already in standard basis coordinates.

$T(\mathbf{b_2}) = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$, which is already in standard basis coordinates.

$T(\mathbf{b_3}) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, which is already in standard basis coordinates.

So our final matrix is $\begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$

$\square$