Worksheet 12

Sections 207 and 319
MATH 54

March 5, 2018

Exercise 1. (a) As a group, discuss the definitions of eigenvector and eigenvalue. Draw pictures!

(b) Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue.

(c) Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(b). To check if $\vec{x}$ is an eigenvector of A, we check to see if $A\vec{x} = \lambda\vec{x}$ for some scalar λ. In this case:

$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector of the given matrix, with corresponding eigenvalue $\lambda = -2$.

(c). 3 is an eigenvalue of A ~~iff~~ and only if the equation $A\vec{x} = 3 \cdot \vec{x}$ has a nontrivial solution. This rearranges to $(A - 3I)\vec{x} = 0$. We check if this has nontriv solutions using augmented matrices!

$\left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ So the system has only the triv solution. So $\lambda = 3$ is <u>not</u> an eigenvalue

Exercise 4. Find a basis for the eigenspace corresponding to each listed eigenvalue.

Exercise 2. Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} \qquad \lambda = 4$$

To find the eigenspace, we find the solution set of $(A-\lambda I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 4 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So the solutions are $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$.

So $\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}$ forms an eigenbasis for the $\lambda = 4$ eigenspace.

Exercise 3. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

The eigenvalues of an $n \times n$ matrix is exactly the set of roots of the characteristic polynomial, $\det(A - \lambda I) = 0$. By thinking about how cofactor expansion works, you can see that this will always be a degree n polynomial, which can have at most n distinct roots.

Exercise 4. Find the characteristic polynomial and the eigenvalues for the matrix.

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$$

The characteristic polynomial of a matrix A is $\det(A-\lambda I)=0$. So we need to find

$$\det\left(\begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix}.$$

Since the middle row is almost all zeros, its easiest to do cofactor expansion on this row:

$$\begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix} = -0\begin{vmatrix} -2 & 3 \\ 7 & -2-\lambda \end{vmatrix} + (1-\lambda)\begin{vmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix} - 0\begin{vmatrix} 5-\lambda & -2 \\ 6 & 7 \end{vmatrix}$$

$$= (1-\lambda)\left[(5-\lambda)(-2-\lambda)-18\right] = (1-\lambda)(\lambda^2-3\lambda-10-18) =$$

$$(1-\lambda)(\lambda^2-3\lambda-28) = (1-\lambda)(\lambda-7)(\lambda+4)$$

So the characteristic polynomial is $(1-\lambda)(\lambda-7)(\lambda+4)=0$

and the eigenvalues are $\lambda=1$, $\lambda=7$, $\lambda=-4$.

Exercise 5. For each of the following matrices, describe in geometric terms the real eigenspaces (if any) and their associated eigenvalues. Do not compute the matrices.

(a) The matrix induced by the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which reflects each vector across the z-axis.

(b) The matrix induced by the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates each vector by $\pi/4$ radians counterclockwise.

For each of these parts, think about which vectors are scaled by T, or in other words "don't change angle" with the origin.

(a). - The z-axis is an eigenspace with eigenvalue 1, since vectors on the z-axis are not affected by the transformation.

- The x-y plane is an eigenspace with eigenvalue -1, since $(x, y, 0)$ is sent to $(-x, -y, 0)$.

(b). There are no real eigenspaces, since no vector is scaled by the transformation.