Worksheet 11

Sections 207 and 219
MATH 54

February 28, 2018

Exercise 1. Find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_\beta$ and the given basis β.

$$\beta = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\} \qquad [\mathbf{x}]_\beta = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

$[\vec{x}]_\beta = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$ means that $\vec{x} = 8\begin{bmatrix} 4 \\ 5 \end{bmatrix} - 5\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Exercise 2. Find the coordinate vector $[\mathbf{x}]_\beta$ of $\mathbf{x}$ relative to the given basis β.

$$\beta = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\} \qquad \mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $[\vec{x}]_\beta = \begin{bmatrix} a \\ b \end{bmatrix}$.

We want to find a, b such that $\begin{bmatrix} -2 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ -3 \end{bmatrix} + b\begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

Thus we need to find solutions corresponding to the augmented matrix $\left[\begin{array}{cc|c} 1 & 2 & -2 \\ -3 & -5 & 1 \end{array}\right]$.

This reduces to: $\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -5 \end{bmatrix}$. So $a = 8$, $b = -5$.

Thus $[\vec{x}]_\beta = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$.

Exercise 3. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be bases for a vector space V, and let $P = [[\mathbf{d}_1]_\mathcal{A}, [\mathbf{d}_2]_\mathcal{A}, [\mathbf{d}_3]_\mathcal{A}]$. Which of the following equations is true for all $\mathbf{x}$ in V?

(a) $[\mathbf{x}]_\mathcal{A} = P[\mathbf{x}]_\mathcal{D}$

(b) $[\mathbf{x}]_\mathcal{D} = P[\mathbf{x}]_\mathcal{A}$

This is the expression for $P_{\mathcal{A} \leftarrow \mathcal{D}}$, so (a) is the correct one.

Exercise 4. Let $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$ and $\mathcal{C} = \{\mathbf{c_1}, \mathbf{c_2}\}$ be bases for $\mathbb{R}^2$. Compute the change of coordinate matrix from $\mathcal{C}$ to $\mathcal{B}$. $\leftarrow$ $P_{\mathcal{B}\leftarrow\mathcal{C}}$

$$b_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, b_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, c_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

To compute this, we row-reduce.

$$\left[\begin{array}{cc|cc} \vec{b}_1 & \vec{b}_2 & \vec{c}_1 & \vec{c}_2 \end{array}\right] = \left[\begin{array}{cc|cc} 7 & -3 & 1 & -2 \\ 5 & -1 & -5 & 2 \end{array}\right]$$

which row reduces to

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{array}\right] \qquad \text{So } P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}.$$

Exercise 5. Mark each statement True or False. Justify each answer.

(a) The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent.

(b) If $V = \mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$, and $\mathcal{C} = \{\mathbf{c_1}, \mathbf{c_2}\}$, then row reduction of $[\mathbf{c_1} c_2 b_1 b_2]$ to $[IP]$ yields a matrix P that satisfies $[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.

True! $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is an invertible matrix (its inverse is $P_{\mathcal{B}\leftarrow\mathcal{C}}$)

An invertible matrix always has lin ind. columns.

False. Row reduction of $\left[\begin{array}{cc|cc} \vec{c}_1 & \vec{c}_2 & \vec{b}_1 & \vec{b}_2 \end{array}\right]$ yields $P_{\mathcal{C}\leftarrow\mathcal{B}}$

and the matrix that satisfies

$[\vec{x}]_{\mathcal{B}} = P[\vec{x}]_{\mathcal{C}}$ is $P_{\mathcal{B}\leftarrow\mathcal{C}}$

Exercise 6. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the standared basis to the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis.

It is actually easier to compute $P_{\mathcal{E} \leftarrow \mathcal{B}}$, so we will do that first.

$$P_{\mathcal{E} \leftarrow \mathcal{B}} = \left[[b_1]_{\mathcal{E}} \; [b_2]_{\mathcal{E}} \; [b_3]_{\mathcal{E}} \right] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

However, the problem asks for $P_{\mathcal{B} \leftarrow \mathcal{E}}$

which is $$\left(P_{\mathcal{E} \leftarrow \mathcal{B}}\right)^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ 3 & -1 & 1 \end{bmatrix}$$

(I didn't show my work for the inverse calculation, let me know if you have questions).

We can now compute

$$[t^2]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{E}} [t^2]_{\mathcal{E}} = \begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

So $t^2 = 3(1 - 3t^2) - 2(2 + t - 5t^2) + (1 + 2t)$.