Worksheet 10

Sections 207 and 219
MATH 54

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Exercise 1. Assume that A is row equivalent to B. Find a basis for the space spanned by the columns of A.

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & 3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We use the echelon form, B, to see that the first, third, and last columns are pivot cols.

So we use these columns of our original matrix, A, as our basis.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$$

Exercise 2. True or false? Give brief justifications.

(a) A linearly independent set in a subspace H is a basis for H.

(b) If a finite set S of nonzero vectors spans a vector space V, then some subsets of S is a basis of V.

(c) If B is an echelon form of a matrix A, the pivot columns of B for a basis of col A.

(d) Every plane in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.

(e) The vector spaces $\mathbb{P}_2$ and $\mathbb{R}^3$ are isomorphic.

(a) False. a lin ind set must also span H in order to be a basis.

(b) True. This is the spanning set thm.

(c) False. You must use the pivot columns of your original matrix A. (See ex. 1 for an example).

(d). False. If a plane does not go through the origin of $\mathbb{R}^3$, it is not even a subspace of $\mathbb{R}^3$, and thus cannot be isomorphic to the vector space $\mathbb{R}^2$.

(e). True! They are both vector spaces of dimension 3. $\{1, x, x^2\}$ serves as a basis for $\mathbb{P}_2$.

Exercise 3. Find a basis for the set of vectors in $\mathbb{R}^3$ on the plane $x + 2y + z = 0$.

This is the same as finding a basis for the solution set of the following augmented matrix:

$$\left[1, 2, 1 \mid 0\right]$$

The solution set in parametric vector form is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So a basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Exercise 4. Using facts that we learned in chapter 1 and 2, explain why a basis of $\mathbb{R}^n$ must contain exactly n vectors.

We first show that it must contain at most n vectors. Suppose we have a set of more than n vectors (say k, where $k > n$). Consider the matrix formed from these vectors as columns: $\underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix}}_{k} \Big\} n$

There are a maximum of n pivots, but k cols, since $k > n$, there cannot be a pivot in every col. So the cols cannot be lin ind. So to be a basis there must be at most n vectors.

We now show that a basis must have at least n vectors. Suppose for contradiction that we have a set of less than n vectors (say k, where $k < n$). Consider the matrix formed from these vectors as columns: $\underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix}}_{k} \Big\} n$

There are a maximum of k pivots, but n rows, $n > k$. So there cannot be a pivot in every row. So the columns do not span $\mathbb{R}^n$, contradicting the fact that our set is a basis. So a basis must have at least n vectors.

Since a basis must have at least and at most n vectors, it must have exactly n vectors.

Exercise 5. Let $T : V \to W$ be a linear transformation. Shot that if $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is linearly dependent V, then $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_p})\}$ is linearly dependent in W. Use this to show that if $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_p})\}$ is linearly independent in W, then $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is linearly independent in V.

Suppose $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly dependent.

Then by def, there is some $c_1, \ldots, c_p$, not all 0, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p = \vec{0}.$$

Taking T of both sides, we get

$$T(c_1\vec{v}_1 + \ldots + c_p\vec{v}_p) = T(\vec{0}).$$

Since T is a lin. transf., this simplifies

$$c_1 T(\vec{v}_1) + \ldots + c_p T(\vec{v}_p) = \vec{0}.$$

Since we know from above that $c_1, \ldots, c_p$ aren't all 0, by definition $\{T(\vec{v}_1), \ldots T(\vec{v}_p)\}$ ~~$T(\vec{v}_1) + \ldots + T(\vec{v}_p)$~~ is a lin dep set.

Suppose $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is lin independent. Suppose for contradiction that $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is lin dep. But then by the previous part, $\{T(\vec{v}_1), \ldots T(\vec{v}_p)\}$ would have to be lin dep, which contradicts our original assumption. So $\{\vec{v}_1, \ldots, \vec{v}_n\}$ must be lin. ind.