Worksheet 6.1 and 6.2

Max's Lecture
MATH 55

July 15, 2019

Exercise A (Various problems from Rosen and Charles' Worksheets). Use counting techniques to compute the following:

1. How many bit strings are there of length eight?
2. How many bit strings of length eight either start with a 1 or end with 2 zeros?
3. Consider the set of integers 1 through 10. How many subsets contain the number 1 or the number 10?
4. How many subsets contain neither the number 1 nor the number 10?
5. Suppose p and q are prime numbers and $n = pq$. How many numbers not exceeding n are relatively prime to n?

1. There are 2^8 bit strings, since we have 2 choices for each bit.

2. There are 2^7 that start with a 1 and 2^6 that end with two zeros. There are 2^5 that both start with a 1 <u>and</u> end with 2 zeros.

So by the subtraction property, the number of strings that start with a 1 <u>or</u> end in two zeros is $2^7 + 2^6 - 2^5$.

3. There are 2^9 subsets that contain the number 1 (since we need to put 1 in the subset and ~~~~ have 2 choices for the other 9 numbers). and similarly there are 2^9 subsets that contain the number 10. There are 2^8 subsets that contain both 1 <u>and</u> 10.

So by the subtraction property, there are $2^9 + 2^9 - 2^8 = 2^{10} - 2^8$ subsets that contain 1 <u>or</u> 10.

4. We subtract the subsets that contain 1 or 10 from the total number of subsets:

$2^{10} - (2^{10} - 2^8) = 2^8$.

5. (on separate page at the end)

Exercise B (Various sources). Answer each question using pigeon hole principle

1. Show that among any group of 5 integers, there are two that have the same remainder when divided by 4.
2. Show that among any group of 3 integers, there are two whose sum is even.
3. How many distinct numbers must be selected from the set of numbers 1 to 6 to guarantee that at least one pair of these numbers add up to 7?
4. There are 5 points inside an equilateral triangle of side length two centimeters. Show that at least two of the points are within 1 centimeter of each other.
5. Challenge: Show that in a group of n people, there are two with an identical number of friends within the group. (no one is friends with themself). Hint: You will need to consider two separate cases.

1. The pigeons are the 5 integers, and the holes are the four possible remainders mod 4. Since there are more pigeons than holes, At least 2 of the pigeons must be in the same hole, So 2 of the integers must have the same remainder.

2. The pigeons are the three integers, and the 2 holes are "being even" or "being odd". Since we have more pigeons than holes, either 2 of the integers are both even or 2 are both odd.

Case 1: We have two integers that are both even. Then their sum is even.

Case 2: We have two integers that are both odd. Then their sum is even.

Either way, we are guaranteed a pair with an even sum.

3. Similarly to the problem we did in class, we can construct 3 holes $\{1,6\}$, $\{2,5\}$, $\{3,4\}$. For each of these, if we pick two numbers in the same hole, they automatically add up to 7. Since we have 3 holes, it sufficies to have four pigeons to guarantee

that two will go in the same hole (and thus add to 7)

4. We divide our triangle into 4 smaller triangles each with side length 1. (these are our holes).

Thus, by pigeon-hole principle two of the dots must go in the same hole.
The maximum distance inside an equilateral triangle with side length 1 is 1. So these two dots must be with 1 centimeter of each other.

5. This one is hard! Our pigeons will be the n people, and our holes will be the possible numbers of friends. Within a group of n people, it is possible to have 0, 1, 2, ... , or n-1 friends. Thus, we have n holes. We cannot use the PHP yet since we have the same number of pigeons and holes. We now use casework:

"= at least one"

Case 1: Suppose everyone has a friend. Thus, since no one has 0 friends, we reduce to the case where we just have n-1 holes, ~~i~~ 1, 2, ... , or n-1 friends.

Since we have ~~pigeo~~ n pigeons and n-1 holes, we have 2 in the same hole. So two people must have the same number of friends as desired

Case 2: One person doesn't have any friends. Call this person A. We have n-1 remaining people, and within this group they can each have 0, ..., n-2 ~~remaining friends~~ friends. However, we have 2 cases again:

Case 2.1: One of the n-1 remaining people has 0 friends. Then this person has the same number of friends as A. So we are done

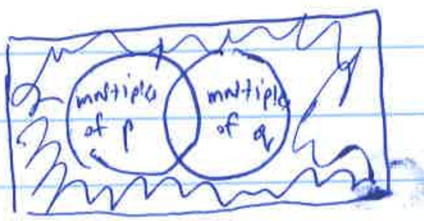
Case 2.2: None of the n-1 remaining people have no friends. So similarly to case 1, there are n-2 holes:

1, 2, ... n-2 and n-1 pigeons

So 2 of the remaining people have to have the same number of friends

Exercise A number 5

Another way of wording this problem is:
How many numbers not exceeding n are multiples of neither p nor q?



So we want to count what is in the shaded portion of this diagram. We can rephrase this as

$$\left|\text{All positive numbers} \le n\right| - \left|\text{numbers that are multiples of } p \text{ or } q\right|.$$

The number of multiples of p or $q \le n$ is

$$\left|\text{numbers} \le n \text{ that are multiples of } p\right| + \left|\text{numbers} \le n \text{ that are multiples of } q\right| - \left|\text{numbers} \le n \text{ that are multiples of both}\right|$$

$$= \frac{n}{p} + \frac{n}{q} - \frac{n}{pq} = q + p - 1.$$

So the total we want is

$$n - (q+p-1) = qp - q - p + 1.$$