Worksheet 2.3 and 2.5

Max's Lecture
MATH 54

July 1, 2019

Scratch work:

$f: A \to B$
$f^{-1}: B \to A$.

correspondence.

Exercise A (Charles wang's worksheet archive, 1.7.25). Let $f : A \to B$ be a one-to-one ~~function~~. What are the compositions $f \circ f^{-1}$ and $f^{-1} \circ f$? i.e. what are their domains and codomains, and do you know another name for these functions?

$f \circ f^{-1}$ goes from B to B. It sends every element $b \in B$ to itself. We call this function I_B.

$f^{-1} \circ f$ goes from A to A. It sends every element $a \in A$ to itself. We call this function I_A.

Exercise B (Charles's worksheet repository). Let A be a set. Consider the set $S = \{f : A \to \{0,1\}\}$ of functions from A to $\{0,1\}$. Can you identify S in terms of a set construction you already know?

This is more of a challenge problem (won't be on any exam) so I won't put a detailed explanation. But: ~~[illegible]~~ we can define a bijection between this set of functions and the power set.

Exercise C . Give an example of a function that is:

1. one-to-one but not onto
2. onto but not one-to-one
3. niether one-to-one nor onto
4. one-to-one and onto

1. $f: \mathbb{R} \to \mathbb{R}$,
   $f(x) = 5^x$.

2. $f: \mathbb{R} \to \mathbb{R}^{\geq 0}$,
   $f(x) = x^2$.

3. $f: \mathbb{R} \to \mathbb{R}$
   $f(x) = x^2$.

4. $f: \mathbb{R} \to \mathbb{R}$
   $f(x) = x + 1$.

Exercise D (Charles worksheets). Show that any subset of a countable set is countable.

Let A be countable, and let $B \subseteq A$.

Since A is countable, we have a bijection $f : A \to S$ between A and some subset of the positive integers $S \subseteq \mathbb{Z}^+$.

Define the restriction of $f : A \to S$ to B (so we are restricting the domain of f to B) as the function $f|_B : B \to S$ defined by $f|_B(b) = f(b)$ for $b \in B$. Then B is in bijection with the image of $f|_B$, which is a subset of a subset of $\mathbb{Z}^+$, so is a subset of $\mathbb{Z}^+$. Hence, B is countable.

(There are other ways of showing this, using casework for example)

Exercise E. (Ritviks worksheet) Find the cardinality of each of the following. Explain.

1. The integers less than 10
2. The integers with absolute value less than 50.
3. The rational numbers
4. The real numbers between 0 and 2.
5. The set $A \times \mathbb{Z}^+$ where A is the set $\{1, 2\}$.

Exercise Challenge problem!. Let A be a nonempty set. Show that there is no surjection from A to its power set.

1. This is countable. It is a subset of integers, and by the previous problem, every subset of a countable set is countable. Since it's not finite, we say it is countably infinite.

2. This is a finite set with cardinality 99. We get this by counting the numbers in the set $\{-49, -48, \ldots, -1, 0, 1, \ldots, 48, 49\}$

3. We showed in class that this is countably infinite.

4. This is uncountable. As part of the proof that $\mathbb{R}$ is countable, we showed that we cannot list real numbers between 0 and 1. Since this is a subset of real numbers between 0 and 2, we definitely cannot list those.

5. This is countably infinite! We can list as follows:
$(1,1), (2,1), (1,2), (2,2), (1,3), (2,3), (1,4), (2,4), \ldots$
(this is similar to showing $\mathbb{Z}$ is countable, we just use 1, 2 instead of $\pm$