1.7-1.8.

Worksheet ~~16-17~~

Max's Lecture
MATH 54

June 26, 2019

In this proof, I use the fact that a^3 is even implies a is even for all integers a.

Exercise A (Charles wang's worksheet archive, 1.7.25). Prove the following by contradiction

1. $\sqrt[3]{2}$ is irrational.
2. At least 10 of any 64 days chosen must fall on the same day of the week.

1. We will prove this by contradiction. Suppose $\sqrt[3]{2}$ is rational. Since we can write every rational number in lowest terms, this means we can write $\sqrt[3]{2} = \frac{p}{q}$, where p, q are integers, $q \neq 0$, and p, q have no common divisors. Cubing both sides, we get $2 = \frac{p^3}{q^3}$ which rearranges to $2q^3 = p^3$. This shows that p^3 is even, so by the note in the upper margin, p is even. Since p is even, we can write $p = 2k$ for some integer k. Subtituting this into $2q^3 = p^3$, we get $2q^3 = (2k)^3 = 8k^3$. So $q^3 = 4k^3 = 2(2k^3)$. So q^3 is even, implying q is even. This contradicts the fact that p, q have no common divisors. So by contradiction, $\sqrt[3]{2}$ must be irrational.

2. We will prove this by contradiction. Suppose out of all of the 64 days, there is no day ~~[illegible]~~ of the week that is chosen more than 9 times. However, this means there is a maximum of $9 \times 7 = 63$ days chosen, since there are 7 days in a week. This contradicts the given premise that there are 64 days chosen. So by contradiction, at least 10 of the days chosen must fall on the same day of the week.

Note: This uses what we call the pigeon hole principle! We will talk more about this later!

Exercise B (1.8.33 and Charles's worksheet repository). Prove each of the proofs by casework:

1. Show that there are no solutions in positive intergers x and y to $x^4 + y^4 = 625$. (Hint: Which cases can you throw out right away?)

2. Prove that if the remainder when dividing n by 3 is 2, that n is not a square. (Hint: Try doing this by contraposition. For another hint, see me!)

1. First of all, $5^4 = 625$, so for all $x \geq 5$, $x^4 \geq 625$.

So if $x \geq 5$ or $y \geq 5$, it is impossible that $x^4 + y^4 = 625$.

Thus, we are left with 16 cases.

x	y
1	1
1	2
1	3
1	4
2	1
2	2
2	3
2	4
3	1
3	2
3	3
3	4
4	1
4	2
4	3
4	4

But WLOG we can just consider the circled cases. (since $x^4 + y^4 = y^4 + x^4$)

If you verify one by one, you will see that none of these pairs give you $x^4 + y^4 = 625$.

So by casework, there are no postive integers x, y such that

$x^4 + y^4 = 625$

2. We will prove by contraposition and casework. Suppose for contraposition that n is a perfect square. So $n = k^2$ for some $k \in \mathbb{Z}^{\geq 0}$. ← (nonnegative integer)

We now show that $n \not\equiv 2 \pmod 3$ by testing 3 cases.

Case 1: $k \equiv 0 \bmod 3$. So $n = k^2 \equiv 0 \pmod 3$

Case 2: $k \equiv 1 \bmod 3$. So $n = k^2 \equiv 1 \pmod 3$

Case 3: $k \equiv 2 \bmod 3$. So $n = k^2 \equiv 4 \equiv 1 \pmod 3$

Note that in all of these cases $n \not\equiv 2 \pmod 3$. Since we have shown the contrapositive is true, the original statement must be true.

Exercise C . Your friend shows you a proof of $|xy| = |x||y|$ for all real x and y. They prove this using the following 4 cases:

1. x, y both nonnegative
2. x nonnegative and y negative
3. x negative and y nonnegative
4. x, y both negative

How could we use the idea of WLOG to shorten the proof.

There is a discussion of this on page 100 of the book. Ask me if you have any questions

Exercise D (1.8.21). Show that if n is an odd integer, then there is a unique integer k such that n is the sum of $k-2$ and $k+3$.

We first show that such an integer k exists, by construction.

Assume n is an odd integer. Thus, there is some integer k such that $n=2k+1$. We now rearrange things a little bit:

$$n = 2k+1 = k+k+1 = k+k+\underbrace{3-2}_{\text{Since } 1=3-2} = (k-2)+(k+3).$$

So we have shown that such an integer k s.t. $n=(k-2)+(k+3)$

We now show uniqueness. Suppose we have 2 integers k, j such that

$n=(k-2)+(k+3)$ and $n=(j-2)+(j+3)$.

Setting these equal to each other, we get:

$$(k-2)+(k+3) = (j-2)+(j+3).$$

Which simplifies to

$$2k+1 = 2j+1.$$

Which simplifies to

$$k=j.$$

So k must be unique.

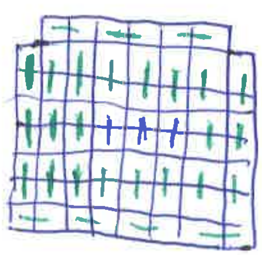
Exercise E. (Example in book, and Ritviks worksheets.) Use these series of problems to practice the process of mathematical exploration! Have fun!

The standard checkerboard is an grid that is 8 squares by 8 squares. A domino is a piece that is 2 squares by 1 square. We say that a board is tiled by dominoes when all the squares are covered with no overlapping dominoes and no dominoes overhanging the edge?

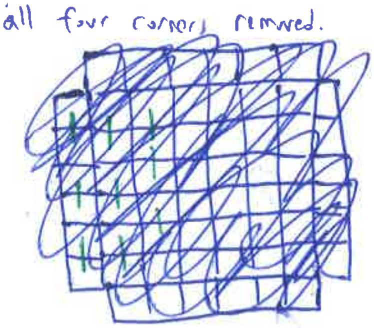
1. Can we tile the standard checkerboard using the dominoes?
2. Try to tile a board obtained by removing one of the four corner squares of a standard checkerboard. Make a conjecture as to whether this is possible.
3. Prove your conjecture.
4. Try to tile a board obtained by removing two opposite corners of the standard checkerboard. Make a conjecture as to whether this is possible.
5. Prove your conjecture. (This is tricky! Ask me for a hint!)
6. Can you use dominoes to tile a standard checkerboard board with 2 adjacent corners removed? What about all 4 corners removed?
7. Prove or disprove that you can use dominoes to tile any rectangular checkerboard with an even number of squares i.e. an m by n board with mn (the number of squares) being even.

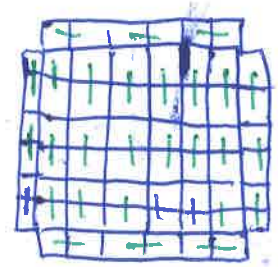
~~[illegible]~~ A discussion of 1-5 is on page

6. It turns out you can tile a standard checkerboard with 2 adjacent corners removed:



You can also tile a checkerboard with all four corners removed.





note: there may be many other ways to tile these.

7. You can always tile any rectangular checkerboard with an even number of squares. Here is a sketch of the proof:

Step 1: Show that if m·n is even, one of m and n must be even.

Step 2: We can now break this into 2 cases:

Case 1: The rectangle has even width. Then, you can tile with horizontal dominoes 

Case 2: The rectangle has even height. Then, you can tile with verticle ~~[illegible]~~ dominoes