Worksheet 1.6-1.7

Max's Lecture
MATH 54

June 26, 2019

Exercise A (1.6.3,1.6.7). What rule of inference is used in each of these argments?

1. Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major. Addition.
2. Jerry is a mathematics major and a computer science major. THerefore, Jerry is a mathematics major. Simplification.
3. If it is rainy, then the pool will be closed. It is rainy. Therefore the pool is closed. Modus ponens
4. If it snows today, the university will close. The university is not closed today. Therefore it did not snow today. Modus tollens.
5. If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, I will sunburn. Therefore, if I go swimming, then I will sunburn. hypothetical syllogism.
6. All men are mortal. Socrates is a man. Therefore, Socratese is mortal.

Universal instantiation.

(You do not need to know these names for any exam).

Exercise B (Example from book). Show that the premises: " If you send me an e-mail, then I will finish writing the program," "If you do not send me an e-mail, then I will go to sleep early,", and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed".

This is worked out on page 77, example 7.

Exercise C (1.6.19). Determine whether each of these arguments is valid. If not, what logical error occurs?

1. If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose $n^2 > 1$. Then $n > 1$.
2. If n is a real number with $n > 3$, then $n^2 > 9$. Suppose $n^2 \leq 9$. Then $n \leq 3$.
3. If n is a real number with $n > 2$, then $n^2 > 4$. Supose $n \leq 2$. Then $n^2 \leq 4$.

1. This argument is not valid. It uses the fallacy of afferming the conclusion.

2. This argument is valid. It uses the contraposition of the initial statement. (or modus tollens)

3. This argument is not valid. It uses the fallacy of denying the hypothesis

(Note: You do not need to know the names of the fallacys for the exam)

Exercise D (Rosen 1.7.1,1.7.7. Find a direct proof of the following:

1. The sum of two odd integers is even.
2. Every odd integer is the difference of two squares. (Hint! Find the difference of k and $k+1$)

1. Let a and b be odd integers. By the definition, there exists $m, n \in \mathbb{Z}$ such that $a = 2m+1$ and $b = 2n+1$.
So $a+b = (2m+1) + (2n+1) = 2m+2n+2 = 2(m+n+1)$.
Since $m+n+1 \in \mathbb{Z}$, by definition $a+b$ is even.

2. Let a be an odd integer. Thus, by definition, there exists a $k \in \mathbb{Z}$ such that $a = 2k+1$. We will now show that
$a = (k+1)^2 - k^2$ since
$(k+1)^2 - k^2 = k^2+2k+1-k^2 = 2k+1 = a$.

Thus, we showed that you can write every odd integer as the difference of two squares.

Exercise E. (Rosen 1.7.19) Show that if n is an integer and $n^3 + 5$ is odd, then n is even. Try doing this in two ways, using a proof by contraposition and a proof by contradiction.

Contraposition

We will first try to prove this by contraposition.
Suppose that n is odd.
Then, there exists $k \in \mathbb{Z}$ s.t. $n = 2k+1$. We now consider n^3+5. Substituting $n=2k+1$, we get:

$(2k+1)^3 + 5 = (8k^3+12k^2+6k+1)+5$

$= 8k^3+12k^2+6k+6 =$

$2(4k^3+6k^2+3k+3)$

Since $4k^3+6k^2+3k+3$ is an integer, by definition n^3+5 is even.

Thus, we have shown the contrapositive of the statement.

So the original statement is true.

Contradiction

We will prove this by contradiction.
Assume that n is integer, n^3+5 is odd, and n is also odd.
Since n is odd, $\exists\, k \in \mathbb{Z}$ s.t. $n=2k+1$.
Plugging this into n^3+5, we get

$n^3+5 = (2k+1)^3+5 =$

$(8k^3+12k^2+6k+1)+5 =$

$8k^3+12k^2+6k+6 =$

$2(4k^3+6k^2+3k+3)$.

This shows n^3+5 is even, contradicting the assumption that n^3+5 is odd.

Since assuming the negation of the original statement led to contradiction, the original statement must be true.

In both methods, we use the same main arguments, they are just framed slightly differently.

Exercise F (Example in book). The following is a famous proof that 1=2. Where is the mistake?

Suppose that $a = b$. We then multiply both sides by a to get $a^2 = ab$. We then subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Factoring both sides, we get $(a-b)(a+b) = b(a-b)$. Then divide both sides by $(a-b)$ to get $(a+b) = b$. Since $a = b$, this is the same as $2b = b$. Dividing by b, we get $2 = 1$.

It can be hard to see, but this argument involves a step in which we divide by 0.

Since we assume that $a=b$, $(a-b)=0$. So the step in which we divide by $a-b$ is not allowed.

Exercise Challenge Problem 1 (rosen 1.7.44). Prove that the following for statements about an integer n are equivalent.

1. n^2 is odd
2. $1 - n$ is even
3. n^3 is odd
4. $n^2 + 1$ is even

(For sake of time, I'm not posting solutions of the challenge problems. Let me know if you have any questions!)