Worksheet 9.1-3

Max's Lecture
MATH 55

July 29, 2019

Exercise A. For every function $f : A \to B$, the set of ordered pairs $(a, f(a))$ is a binary relation from A to B.

1. Give an example of such a relation.
2. Are there relations that cannot be expressed in this way?

1. one example is the relation on $\mathbb{R}$ consisting of ordered pairs (x,y) where $y=x^2$.

2. Yes! For example: the relation R on $\mathbb{R}$ where
$(x,y) \in R$ iff $x \leq y$.
$(1,2)$ and $(1,3)$ are both in the relation, but it is impossible for a function to assign 1 to both 2 and 3
(goes against the definition of a function)

I like to think of relations as kind of a generalization of functions. We are still pairing elements together but with less restrictions

Let S be this set.

Exercise B. 1. How many relations are there on a set of n elements?

2. How many reflexive relations are there?

3. How many symmetric relations are there?

1. 2^{n^2}. This is because there are n^2 ordered pairs in $S \times S$. For each ordered pair, we have 2 options: whether or not to put it in the relation. So there are $\boxed{2^{n^2}}$ ways to build a relation.

2. We know that all the ordered pairs of the form (x,x) have to be in R. There are n of these. Thus, there are n^2-n choices we still have to make. (as in decide whether or not the ordered pair is in R). So there are $\boxed{2^{n^2-n}}$ ways to build such a relation.

3. For $x \neq y$, if (x,y) is in R so is (y,x). So instead of choosing which ordered pairs are in the relation, we choose which unordered pairs are in the relation. There are $2^{\binom{n}{2}}$ to do this.
We also have to choose which of the pairs of the form (x,x) are in R. there are 2^n ways to do this.

Putting this together, we have

$$\boxed{2^n 2^{\binom{n}{2}} \text{ symmetric relations} = 2^{n+\binom{n}{2}}}$$

Exercise C. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisemmetric, and/or transitive, where $(x, y) \in R$ if and only if:

1. $x \neq y$ symmetric.
2. $xy \geq 1$ symetric, transitive
3. $x = y + 1$ or $x = y - 1$ symmetric.
4. $x \equiv y (\text{mod } 7)$ sym, reflexive, transitive
5. $y = x^2$ antisymmetric.

I will justify #2, let me know if you have questions about any of the others.

or base set

The domain is very important here.

This relation is symmetric because since $xy = yx$, if $xy \geq 1$ then $yx \geq 1$.

This relation is transitive because since the domain is ~~positive~~ integers, $xy \geq 1$ precisely when x, y are both positive or both negative. If $(x,y), (y,z) \in R$, x,y are both positive or both negative, y,z are both positive or both negative, so x,z are both positive or both negative. Thus, $(x,z) \in R$.

This relation is <u>not</u> antisymmetric since $(1,3)$, $(3,1)$ are both in R, but $1 \neq 3$.

This relation is not transitive since $(0,0) \notin R$.