

# Worksheet 8.4-6

Max's Lecture  
MATH 55

July 25, 2019

**Exercise A.** Give an expression for the generating function of the number of ways to distribute  $n$  identical toys to 3 children such that each child gets at least 2 toys.

On prev. work sheet.

**Exercise B.** All the sets considered below are finite:

1. Write a formula for  $|A \cup B|$  in terms of  $|A|, |B|, |A \cap B|$ .
2. Write a formula for  $|A \cup B \cup C|$  in terms of  $|A|, |B|, |C|, |A \cap B|, |B \cap C|, |A \cap C|, |A \cap B \cap C|$ .
3. Write a similar formula for  $|A \cup B \cup C \cup D|$ .
4. Do you notice any pattern?

$$1. |A \cup B| = \cancel{|A| + |B|} |A| + |B| - |A \cap B|$$

$$2. |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$3. |A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| \\ - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| \\ + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

4. The idea is to conjecture the formula for inclusion/exclusion 😊

**Exercise C.** How many elements are in the union of four sets if the sets have 50, 60, 70, and 80 elements respectively, each pair of the sets has 5 elements in common, each triple of the sets has 1 element in common, and no element is in all 4 sets?

By inclusion/exclusion (or part 3 of the previous exercise), we have.

$$|\text{union}| = 50 + 60 + 70 + 80 - \binom{4}{2} 5 + \binom{4}{3} \cdot 1 - 0.$$

$$= 50 + 60 + 70 + 80 - 6 \cdot 5 + 3.$$

$$= 227 \text{ (I think not very good at subtracting haha)}$$

Exercise D. Use the principle of inclusion exclusion to find the number of solutions to  $x_1 + x_2 + x_3 = 11$  with  $x_1 \leq 3$  and  $x_2 \leq 4$  and  $x_3 \leq 6$ .

We do this by subtracting the solutions when

$$A) x_1 \geq 4 \quad \underline{OR}$$

$$B) x_2 \geq 5 \quad \underline{OR}$$

$$C) x_3 \geq 7$$

So our final answer is Total # nonneg integer solutions -  $|A \cup B \cup C|$

$$\text{Total \# of solutions} = \binom{11+3-1}{11} = 78.$$

$$|A| = \binom{3+7-1}{7} = 36$$

$$|B| = \binom{3+6-1}{6} = 28$$

$$|C| = \binom{3+4-1}{4} = 15$$

$$|A \cap B| = \binom{3+2-1}{2} = 6$$

$$|A \cap C| = \binom{3+0-1}{0} = 1.$$

$$|B \cap C| = 0 \text{ since } 5+7 > 11$$

$$|A \cap B \cap C| = 0 \text{ since } 4+5+7 > 11.$$

So our desired number is

$$78 - (36 + 28 + 15 - 6 - 1 - 0 + 0) =$$

6.