# Worksheet 1.1-1.5

Max's Lecture
MATH 54

June 25, 2019

**Exercise A.** Recall the de Morgen's laws:

(a) $\neg(p \vee q) = \neg p \wedge \neg q$

(b) $\neg(p \wedge q) = \neg p \vee \neg q$

Convince yourself that these are true, either by using a truth table, a venn diagram, or some other argument.

I will do this with venn diagrams, but there are many other ways to do it.

(a). We will first take a few steps to draw a venn diagram for $\neg(p \vee q)$.

First, recall that the venn diagram for $p \vee q$ is

$p \vee q$

To get the negation of this, we swap the shading

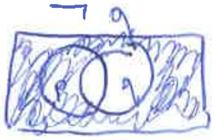
$\neg(p \vee q)$

We now take a few steps to draw a venn diagram for $\neg p \wedge \neg q$:

The venn diagrams for $\neg p$ and $\neg q$ are:

$\neg p$

$\neg q$



To get the venn diagram for $\neg p \wedge \neg q$, we shade the regions that are shaded in Both of the above diagrams.

So we get $\neg p \wedge \neg q$

Since they have the same venn diagrams, $\neg p \wedge \neg q$ and $\neg(p \vee q)$ are log. eq.

(b) We will now take a few steps to draw a venn diagram for $\neg(p \wedge q)$.

$p \wedge q$ is given by

so $\neg(p \wedge q)$ is

We now take a few steps to draw the diagram for $\neg p \vee \neg q$.

~~this diagram~~ To get this, we shade all the regions that are shaded in at least one of the diagrams for $\neg p$, $\neg q$.

$\neg p \vee \neg q$

So we see that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ have the same diagram and are thus log. eq.

**Exercise B (1.2.39 from Rosen).** A detective has interviewed four witnesses to a crime. From the stories of the witnesses the detective has concluded that if the butler is telling the truth then so is the cook; the cook and the gardener cannot both be telling the truth; the gardener and the handyman are not both lying, and if the handyman is telling the truth then the cook is lying. For each of the four witnesses, can the detective determine whether that person is telling the truth or lying? Explain your reasoning.

~~In this write-up, we will represent~~

In this write-up, I will use the following shorthand:

$b$: "butler is telling the truth"

$c$: "cook is telling the truth"

$g$: "gardner is telling the truth."

$h$: "handyman is telling the truth".

We can write the information we are given as compound props.

- ☆ $b \to c$
- ☆ $\neg(c \wedge g)$ which can be rewritten as $\neg c \vee \neg g$
- ☆ $\neg(\neg g \wedge \neg h)$ which can be rewritten as $g \vee h$.
- ☆ $h \to \neg c$

So we want to find the truth values of $b, c, g, h$ that make

$$(b \to c) \wedge (\neg c \vee \neg g) \wedge (g \vee h) \wedge (h \to \neg c)$$

true.

~~One way to do this is through truth tables.~~

~~Other methods include~~ ~~One way to do this is through truth tables. Other methods exist~~

In class, we shared how to do this using truth tables. (ask me if you have any questions)

So here, I will sketch how to do this using case work.

To start off, we have two cases; either the butler is telling the truth or lying (but not both).

Suppose the butler is truthful. Since $b \to c$, the cook is also truthful. Since $(\neg c \vee \neg g)$, the gardener must be lying. Since $(g \vee h)$, the handyman must be telling the truth. Since $h \to \neg c$, the cook must be lying. But this contradicts our earlier conclusion that the cook is truthful. So our assumption that the butler is truthful must have been false.

So we can conclude that the butler is lying.

From a similar analysis about the cook, we can conclude the cook is lying.

Now consider the gardener. It turns out that assuming $g$ is true does not lead to contradiction. It also turns out that assuming $g$ is false does not lead to contradiction. So we cannot conclude whether or not the gardener is lying.

The same thing happens for the handyman.

So in conclusion, we know that the butler and the cook are lying, but we do not know anything about the handyman or the gardener.

**Exercise C (from Ritviks worksheet archive).** Find a compound proposition involving $p, q, r$ that is true when exactly two of $p, q, r$ are true and false otherwise.

There are 3 options:

- $p, q$ are both true and $r$ is false.
- $p, r$ are both true and $q$ is false
- $q$ and $r$ are both true and $p$ is false.

These can be rewritten using symbols as:

- $p \wedge q \wedge \neg r$
- $p \wedge \neg q \wedge r$
- $\neg p \wedge q \wedge r$.

Since we just need at least one of these three conditions to be true in order to satisfy the requirements, we put these three pieces together with $\vee$ in between.

$$(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r).$$

**Exercise D (Rosen 1.5.15.** Determine the truth value of each of these statemennts if the domain for all variables consists of integers:

| | integers | real numbers |
|---|---|---|
| 1. $\forall n(n^2 \geq 0)$ | T | T |
| 2. $\exists n(n^2 = 2)$ | F | T |
| 3. $\forall n(n^2 > n)$ | F ($1^2 \not> 1$) | F. (for same reason) |
| 4. $\exists n(n^2 < 0)$ | F | F |

Do your answers change if you let the domain be all real numbers? Two of these statements are negations of each others. Which ones?

1 and 4 are negations of each other.

(You can see this by using demorgans laws for quantifiers.)

I should have said that domain is all animals.

**Exercise E.** Let $C(x)$ be the statement "x is a cat" and $F(x)$ be the statement "x is fluffy". Write the following statements in English:

1. $\exists x(C(x) \rightarrow F(x))$
2. $\exists x(C(x) \wedge F(x))$

Do these mean the same thing? Why or why not?

~~1.) There exists an cat such that if it.~~

1. There exists an animal such that if it is a cat, it is fluffy.

2. There exists an animal that is a cat and is fluffy.
   If you want, you can write this more simply as "There exists a fluffy cat".

These statements are different! (tho at first the difference may be hard to see).

For 1, this prop is true if the domain has an animal that is not a cat in it.

This ~~example~~ animal does not work as an example to show that 2. is true.

**Exercise F (modified Rosen 1.5.10).** Let $F(x, y)$ be the statement "x can fool y". Use quantifiers to express each of these statements:

1. Evelyn can fool everybody.
2. Everyone can be fooled by somebody.
3. There is somebody who can fool everybody.
4. Challenge: Nancy can fool exactly two people.

Now negate each of these sentence both using logical notatation and then translate to English. Do not just say "it is not the case that blah blah blah".

1. $\forall y \; F(\text{evelyn}, y)$.

2. $\forall y \; \exists x \; F(x, y)$.

3. $\exists x \; \forall y \; F(x, y)$.

4. This is harder. We need to specify a bunch of things:
   - There are two people, x, y, who nancy can fool.
   - x and y are not the same person.
   - if nancy can fool z, z must be one of x or y (since then nancy would be able to fool three people)

   Putting this all together, we get:

   ~~$\exists x \exists y \left( F(\text{nancy}, x) \wedge F(\text{nancy}, y) \wedge x \neq y \wedge (F(\text{Nancy}, z) \to z = x \vee z = y) \right)$~~

$$\exists x \exists y \left( F(\text{nancy}, x) \wedge F(\text{nancy}, y) \wedge x \neq y \wedge \forall z \left( F(\text{Nancy}, z) \to (z = x \vee z = y) \right) \right)$$

Note: There may be a different way to write this.

**Exercise G (Rosen 1.5.21).** Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that every positive integer is the sum of the squares of four integers.

The domain is all integers.

$$\forall a > 0 \; \exists b \exists c \exists d \exists e \; (a = b^2 + c^2 + d^2 + e^2).$$

**Exercise Challenge 1 (from Ritvik's worksheet archive).** Consider an island with two kinds of inhabitants: Knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people, A and B. Determine, if possible, what A and B are if they address you in the ways described. If you cannot determine what these two people are, can you draw any conclusions?

1. A says at least one of us is a knave and B says nothing
2. A says "the two of us are both knights" and B says "A is a knave"

$\boxed{1.}$ We do some casework. Suppose A is a knight. Then the statement "At least one of us is a knave" is true, so B must be a knave. There are no ~~possibilities~~ contradictions, so it is possible that A is a knight.

Now suppose A is a knave. Then A's statement is false, so neither of A, B can be knaves. This contradicts the fact that A is a knave.

Since Case 2 is impossible, it must be the case that A is a knight, implying B is a knave.

$\boxed{2}$ We again do some casework.

Case 1: ~~A is a knight, B~~ Suppose A is a knight. The A's statement must be true, so B must also be a knight. Then B's statement must be true, contradicts the assumption that A is a knight. So it is not possible for A to be a knight.

Case 2: Suppose A is a knave. Then A's statement must be false (but this gives us no further information). Since A is a knave, B's statement is true. So B must be a knight.

Case 2 is the only possible case, so we conclude A is a knave, B is a knight.

**Exercise Challenge 2.** It is a fact that every compound proposition (that perhaps uses implications, biconditonals, or logical operators that we have not learned yet) is logically equivalent to a compound proposition that uses only $\neg, \vee, \wedge$. Can you explain why this is true?

Every compound prop. can be described using a truth table, and every truth table can be written down using $\neg, \vee, \wedge$. For example

| $p$ | $q$ | $p$ ☺ $q$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | T |
| F | F | T |

can be described as $(\neg p \wedge q) \vee (\neg p \wedge \neg q)$.